Section 6.1 Slope Fields and Euler's Method

- Use initial conditions to find particular solutions of differential equations.
- Use slope fields to approximate solutions of differential equations.
- Use Euler's Method to approximate solutions of differential equations.

General and Particular Solutions

In this text, you will learn that physical phenomena can be described by differential equations. In Section 6.2, you will see that problems involving radioactive decay, population growth, and Newton's Law of Cooling can be formulated in terms of differential equations.

A function $y = f(x)$ is called a **solution** of a differential equation if the equation is satisfied when y and its derivatives are replaced by $f(x)$ and its derivatives. For example, differentiation and substitution would show that $y = e^{-2x}$ is a solution of the differential equation $y' + 2y = 0$. It can be shown that every solution of this differential equation is of the form

$$y = Ce^{-2x} \qquad \text{General solution of } y' + 2y = 0$$

where C is any real number. This solution is called the **general solution.** Some differential equations have **singular solutions** that cannot be written as special cases of the general solution. However, such solutions are not considered in this text. The **order** of a differential equation is determined by the highest-order derivative in the equation. For instance, $y' = 4y$ is a first-order differential equation.

NOTE First-order linear differential equations are discussed in Section 6.4.

In Section 4.1, Example 8, you saw that the second-order differential equation $s''(t) = -32$ has the general solution

$$s(t) = -16t^2 + C_1t + C_2 \qquad \text{General solution of } s''(t) = -32$$

which contains two arbitrary constants. It can be shown that a differential equation of order n has a general solution with n arbitrary constants.

EXAMPLE 1 Verifying Solutions

Determine whether the function is a solution of the differential equation $y'' - y = 0$.

a. $y = \sin x$ **b.** $y = 4e^{-x}$ **c.** $y = Ce^x$

Solution

a. Because $y = \sin x$, $y' = \cos x$, and $y'' = -\sin x$, it follows that

$$y'' - y = -\sin x - \sin x = -2 \sin x \neq 0.$$

So, $y = \sin x$ is *not* a solution.

b. Because $y = 4e^{-x}$, $y' = -4e^{-x}$, and $y'' = 4e^{-x}$, it follows that

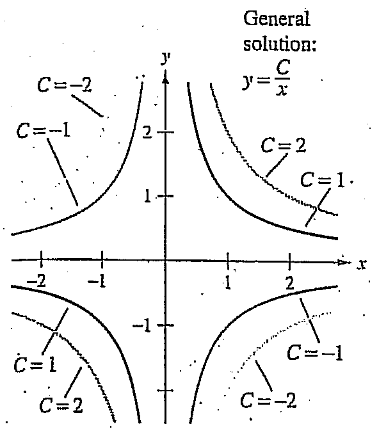
$$y'' - y = 4e^{-x} - 4e^{-x} = 0.$$

So, $y = 4e^{-x}$ is a solution.

c. Because $y = Ce^x$, $y' = Ce^x$, and $y'' = Ce^x$, it follows that

$$y'' - y = Ce^x - Ce^x = 0.$$

So, $y = Ce^x$ is a solution for any value of C.



Solution curves for $xy' + y = 0$
Figure 6.1

Geometrically, the general solution of a first-order differential equation represents a family of curves known as **solution curves,** one for each value assigned to the arbitrary constant. For instance, you can verify that every function of the form

$$y = \frac{C}{x} \qquad \text{General solution of } xy' + y = 0$$

is a solution of the differential equation $xy' + y = 0$. Figure 6.1 shows four of the solution curves corresponding to different values of C.

As discussed in Section 4.1, **particular solutions** of a differential equation are obtained from **initial conditions** that give the value of the dependent variable or one of its derivatives for a particular value of the independent variable. The term "initial condition" stems from the fact that, often in problems involving time, the value of the dependent variable or one of its derivatives is known at the *initial* time $t = 0$. For instance, the second-order differential equation $s''(t) = -32$ having the general solution

$$s(t) = -16t^2 + C_1t + C_2 \qquad \text{General solution of } s''(t) = -32$$

might have the following initial conditions.

$$s(0) = 80, \quad s'(0) = 64 \qquad \text{Initial conditions}$$

In this case, the initial conditions yield the particular solution

$$s(t) = -16t^2 + 64t + 80. \qquad \text{Particular solution}$$

EXAMPLE 2 Finding a Particular Solution

For the differential equation $xy' - 3y = 0$, verify that $y = Cx^3$ is a solution, and find the particular solution determined by the initial condition $y = 2$ when $x = -3$.

Solution You know that $y = Cx^3$ is a solution because $y' = 3Cx^2$ and

$$\begin{aligned} xy' - 3y &= x(3Cx^2) - 3(Cx^3) \\ &= 0. \end{aligned}$$

Furthermore, the initial condition $y = 2$ when $x = -3$ yields

$$y = Cx^3 \qquad \text{General solution}$$

$$2 = C(-3)^3 \qquad \text{Substitute initial condition.}$$

$$-\frac{2}{27} = C \qquad \text{Solve for } C.$$

and you can conclude that the particular solution is

$$y = -\frac{2x^3}{27}. \qquad \text{Particular solution}$$

Try checking this solution by substituting for y and y' in the original differential equation.

NOTE To determine a particular solution, the number of initial conditions must match the number of constants in the general solution.

---

indicates that in the HM mathSpace® CD-ROM *and the online* Eduspace® *system for this text, you will find an Open Exploration, which further explores this example using the computer algebra systems* Maple, Mathcad, Mathematica, *and* Derive.

Exercises for Section 6.1

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–8, verify the solution of the differential equation.

Solution	*Differential Equation*
1. $y = Ce^{4x}$	$y' = 4y$
2. $y = e^{-x}$	$3y' + 4y = e^{-x}$
3. $x^2 + y^2 = Cy$	$y' = 2xy/(x^2 - y^2)$
4. $y^2 - 2\ln y = x^2$	$\dfrac{dy}{dx} = \dfrac{xy}{y^2 - 1}$
5. $y = C_1 \cos x + C_2 \sin x$	$y'' + y = 0$
6. $y = C_1 e^{-x}\cos x + C_2 e^{-x}\sin x$	$y'' + 2y' + 2y = 0$
7. $y = -\cos x \ln\lvert\sec x + \tan x\rvert$	$y'' + y = \tan x$
8. $y = \frac{2}{3}(e^{-2x} + e^x)$	$y'' + 2y' = 2e^x$

In Exercises 9–12, verify the particular solution of the differential equation.

Solution	*Differential Equation and Initial Condition*
9. $y = \sin x \cos x - \cos^2 x$	$2y + y' = 2\sin(2x) - 1$ $y\left(\dfrac{\pi}{4}\right) = 0$
10. $y = \frac{1}{2}x^2 - 4\cos x + 2$	$y' = x + 4\sin x$ $y(0) = -2$
11. $y = 6e^{-2x^2}$	$y' = -4xy$ $y(0) = 6$
12. $y = e^{-\cos x}$	$y' = y\sin x$ $y\left(\dfrac{\pi}{2}\right) = 1$

In Exercises 13–18, determine whether the function is a solution of the differential equation $y^{(4)} - 16y = 0$.

13. $y = 3\cos x$
14. $y = 3\cos 2x$
15. $y = e^{-2x}$
16. $y = 5\ln x$
17. $y = C_1e^{2x} + C_2e^{-2x} + C_3\sin 2x + C_4\cos 2x$
18. $y = 3e^{2x} - 4\sin 2x$

In Exercises 19–24, determine whether the function is a solution of the differential equation $xy' - 2y = x^3e^x$.

19. $y = x^2$
20. $y = x^2e^x$
21. $y = x^2(2 + e^x)$
22. $y = \sin x$
23. $y = \ln x$
24. $y = x^2e^x - 5x^2$

In Exercises 25–28, some of the curves corresponding to different values of C in the general solution of the differential equation are given. Find the particular solution that passes through the point shown on the graph.

Solution	*Differential Equation*
25. $y = Ce^{-x/2}$	$2y' + y = 0$
26. $y(x^2 + y) = C$	$2xy + (x^2 + 2y)y' = 0$
27. $y^2 = Cx^3$	$2xy' - 3y = 0$
28. $2x^2 - y^2 = C$	$yy' - 2x = 0$

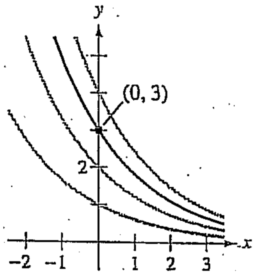


Figure for 25

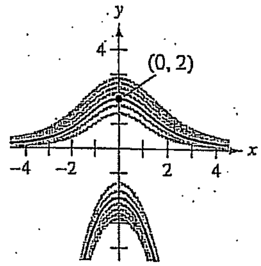


Figure for 26

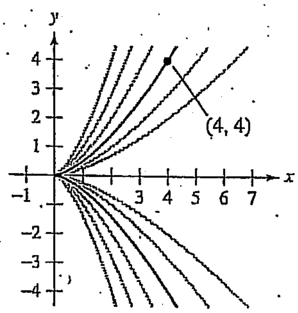


Figure for 27

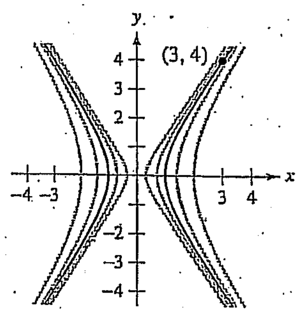


Figure for 28

In Exercises 29 and 30, the general solution of the differential equation is given. Use a graphing utility to graph the particular solutions for the given values of C.

29. $4yy' - x = 0$
$4y^2 - x^2 = C$
$C = 0, C = \pm 1, C = \pm 4$

30. $yy' + x = 0$
$x^2 + y^2 = C$
$C = 0, C = 1, C = 4$

In Exercises 31–36, verify that the general solution satisfies the differential equation. Then find the particular solution that satisfies the initial condition.

31. $y = Ce^{-2x}$
$y' + 2y = 0$
$y = 3$ when $x = 0$

32. $3x^2 + 2y^2 = C$
$3x + 2yy' = 0$
$y = 3$ when $x = 1$

33. $y = C_1\sin 3x + C_2\cos 3x$
$y'' + 9y = 0$
$y = 2$ when $x = \pi/6$
$y' = 1$ when $x = \pi/6$

34. $y = C_1 + C_2\ln x$
$xy'' + y' = 0$
$y = 0$ when $x = 2$
$y' = \frac{1}{2}$ when $x = 2$