Dawson College
Mathematics Department

WINTER 2021 Final Examination (Solutions)

Remedial Activities For Secondary V Mathematics (201 – 015 – RE)

27/05/2021 18:30- 21:30 (3h)

Instructor(s): Pavel S.

Section(s): 01

Last Name: ______________________

First Name: ______________________

ID: ______________________

This exam contains 17 pages (including this cover page) and 16 problems. Check to see if any pages are missing. The following rules apply:

- **You may *not* use your books or any other notes**.
- You are allowed to use a calculator *or* other material on this exam within the regulations of the college or permitted by the teacher.
- **You are required** to show your work on each problem on this exam.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering could receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- **A formula sheet is attached at the end of this exam**.
- **This exam is worth 50% or 100% of your final grade. Failing the exam with less than 50% will yield in automatic failure of the course with a maximum overall grade of 55%**
- **Good luck!**

Do not write on this page except for the information to be filled (upper right corner).

Problem	Points	Score
1	4	
2	5	
3	5	
4	5	
5	4	
6	4	
7	17	
8	8	
9	4	
10	12	
11	7	
12	4	
13	10	
14	3	
15	6	
16	2	
Total:	100	

Question 1 (*4 points*)

Divide using long division:

$$\frac{x^4 + 4x^3 - 2x + 3}{x^2 + x}$$

Solution:

$$\begin{array}{rl}
 & x^2 + 3x - 3 \\
x^2 + x \big) & x^4 + 4x^3 \qquad - 2x + 3 \\
 & \underline{- x^4 \quad - x^3} \\
 & 3x^3 \\
 & \underline{- 3x^3 - 3x^2} \\
 & - 3x^2 - 2x \\
 & \underline{3x^2 + 3x} \\
 & x + 3
\end{array}$$

Hence,

$$\frac{x^4 + 4x^3 - 2x + 3}{x^2 + x} = x^2 + 3 - 3 + \frac{x + 3}{x^2 + x}$$

Question 2 (*5 points*)

Perform the following operation and simplify your answers:

$$\frac{2x^2-18}{x^2+x-12} \div \frac{2x^2-2}{x^2+3x-4}$$

Hence, the answer is

$$x^2+3x-3+\frac{x+3}{x^2+x}$$

Solution:

$$\begin{aligned}
\frac{2x^2-18}{x^2+x-12} \div \frac{2x^2-2}{x^2+3x-4} &= \frac{2x^2-18}{x^2+x-12} \cdot \frac{x^2+3x-4}{2x^2-2} \\
&= \frac{2(x^2-9)}{(x+4)(x-3)} \cdot \frac{(x+4)(x-1)}{2(x^2-1)} \\
&= \frac{\cancel{2}\cancel{(x-3)}(x+3)}{\cancel{(x+4)}\cancel{(x-3)}} \cdot \frac{\cancel{(x+4)}\cancel{(x-1)}}{\cancel{2}\cancel{(x-1)}(x+1)} \\
&= \frac{x+3}{x+1}
\end{aligned}$$

Question 3 (*5 points*)

Rationalize the denominator of the expression below:

$$\frac{3}{3-\sqrt{6}}$$

Solution:

$$\frac{3}{3-\sqrt{6}} \cdot \frac{3+\sqrt{6}}{3+\sqrt{6}} = \frac{3(3+\sqrt{6})}{9-6} = \frac{3(3+\sqrt{6})}{3} = 3+\sqrt{6}$$

Question 4 (*5 points*)

Simplify the expression below

$$\frac{\left(a^{-2}b^{3}\right)^{-4}}{(a^{-3}b^{2})(ab)^{3}}$$

Solution:

$$\begin{aligned}\frac{\left(a^{-2}b^{3}\right)^{-4}}{(a^{-3}b^{2})(ab)^{3}} &= \frac{a^{8}b^{-12}}{(a^{-3}b^{2})a^{3}b^{3}} \\ &= \frac{a^{8}b^{-12}}{b^{5}} \\ &= a^{8}b^{-17} \\ &= \frac{a^{8}}{b^{17}}\end{aligned}$$

Question 5 (*4 points*)

Solve the inequality and graph the solution

$$-1 < \frac{(3-x)}{5} \leq 4$$

Solution:

$$\begin{aligned} -1 &< \frac{(3-x)}{5} \leq 4 \\ -5 &< 3-x \leq 20 \\ -8 &< -x \leq 17 \\ 8 &> x \geq -17 \end{aligned}$$

That is, $-17 \leq x < 8$

Question 6 (*4 points*)

Solve the following linear system by the method of elimination.

$$\begin{cases} 6x - 2y = 3 \\ 5x - 5y = 10 \end{cases}$$

Solution:
By multiplying the second equation by -2/5, the system is equivalent to

$$\begin{cases} 6x - 2y = 3 \\ -2x + 2y = -4 \end{cases}$$

So adding the two equation, we obtain

$$4x = -1$$

or $x = -1/4$. Using any of the two equations, say the first, with the value of x, we find

$$6\left(-\frac{1}{4}\right) - 2y = 3 \Rightarrow -\frac{3}{4} - y = \frac{3}{2} \Rightarrow -y = \frac{3}{2} + \frac{3}{4} \Rightarrow y = -\frac{9}{4}$$

Question 7 (*17 points*)

Solve for x in the following equations by the method of your choice unless specified otherwise:

(5 points) (a) $\sqrt{4x} - x + 3 = 0$

Solution:

$$\begin{aligned} \sqrt{4x} - x + 3 &= 0 \\ \sqrt{4x} &= x - 3 \\ 4x &= (x-3)^2 \\ 4x &= x^2 - 6x + 9 \\ x^2 - 10x + 9 &= 0 \\ (x-9)(x-1) &= 0 \end{aligned}$$

Hence, by the Zero Factor Property, $x = 1$ or $x = 9$. Observe that $x = 1$ does not solve the original equation (i.e., $2 - 1 + 3 \neq 0$) while $x = 9$ does. So, the only solution is $x = 9$.

(3 points) (b) $3x^2 + 8 = 8x$

Solution:

$$\begin{aligned} 3x^2 + 8 &= 8x \\ 3x^2 - 8x + 8 &= 0 \end{aligned}$$

Appealing to the Quadratic Formula:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-8) \pm \sqrt{(-8)^2 - 4(3)(8)}}{2(3)} \\ &= \frac{8 \pm \sqrt{64 - 96}}{6} \end{aligned}$$

Since $64 - 96 < 0$, there are no solutions.

(5 points) (c) $\dfrac{x-3}{x^2-1}+\dfrac{3}{x-1}=\dfrac{6}{x+1}$

Solution: We will start by factorizing the denominators, finding the L.C.D., and multiplying by it:

$$\frac{x-3}{x^2-1}+\frac{3}{x-1}=\frac{6}{x+1}$$
$$\frac{x-3}{(x-1)(x+1)}+\frac{3}{x-1}=\frac{6}{x+1}$$

The L.C.D. is $(x-1)(x+1)$; hence, when we multiply by it we obtain

$$\begin{aligned} x-3+3(x+1) &= 6(x-1) \\ x-3+3x+3 &= 6x-6 \\ -2x &= -6 \\ x &= 3 \end{aligned}$$

Observe that none of the original denominators are zero for $x=3$

(4 points) (d) $5^{3x+1}=6^{x-1}$

Solution: Using ln:

$$\begin{aligned} 5^{3x+1} &= 6^{x-1} \\ \ln 5^{3x+1} &= \ln 6^{x-1} \\ (3x+1)\ln 5 &= (x-1)\ln 6 \\ 3x\ln 5+\ln 5 &= \ln 6x-\ln 6 \\ 3x\ln 5-\ln 6x &= -\ln 6-\ln 5 \\ x(3\ln 5-\ln 6) &= -\ln 6-\ln 5 \\ x &= \frac{-\ln 6-\ln 5}{3\ln 5-\ln 6} \\ &= \frac{\ln 6+\ln 5}{\ln 6-3\ln 5} \\ &= \frac{\ln(6\cdot 5)}{\ln 6-\ln 5^3} \\ &= \frac{\ln(6\cdot 5)}{\ln(6/5^3)} \end{aligned}$$

Question 8 (*8 points*)

Consider the points $A(-3,4)$ and $B(-5,-2)$.

(4 points) (a) Find the equation of the line that passes through the points A and B.

Solution: We want to find $y = ax + b$, we start by finding a:

$$a = \frac{\Delta y}{\Delta x} = \frac{-2-4}{-5-(-3)} = \frac{-6}{-2} = 3$$

To find b, we use either A or B, say A, then

$$\begin{aligned} 4 &= 3(-3) + b \\ 4 &= -9 + b \\ b &= 13 \end{aligned}$$

And the equation of the line is $y = 3x + 13$.

(2 points) (b) Find the midpoint between the points A and B.

Solution:

$$\mathcal{M} = \left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}\right) = \left(\frac{-3-5}{2}, \frac{4-2}{2}\right) = (-4, 1)$$

(2 points) (c) Find the distance between the points A and B.

Solution:

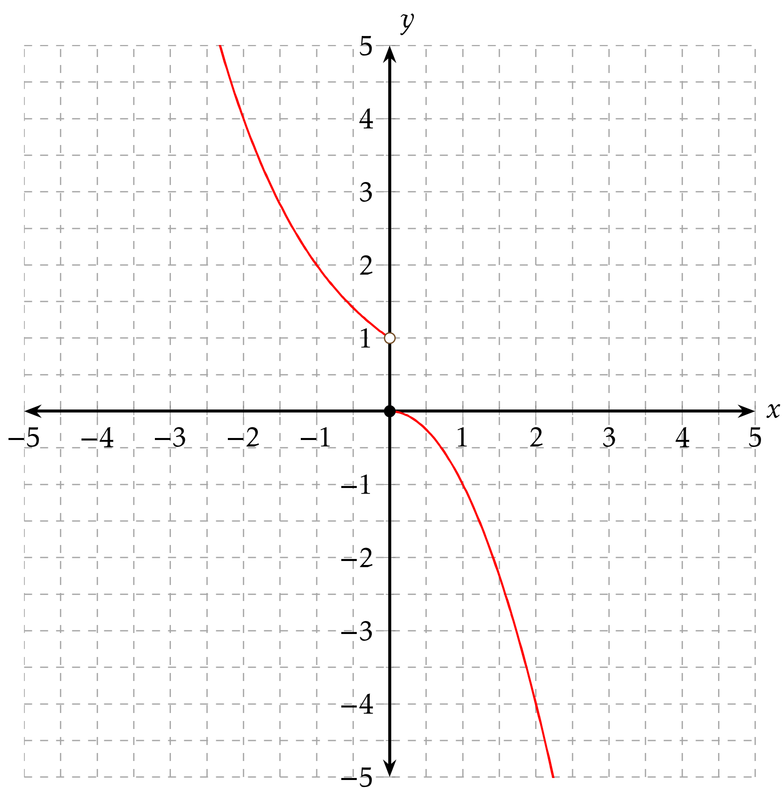
$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(-5-(-3))^2 + (-2-4)^2} \\ &= \sqrt{(-2)^2 + (-6)^2} \\ &= \sqrt{4 + 36} \\ &= \sqrt{40} \\ &= 2\sqrt{10} \end{aligned}$$

Question 9 (*4 points*)

Graph the function below. State its range and domain:

$$f(x) = \begin{cases} \left(\frac{1}{2}\right)^x & , x < 0 \\ -x^2 & , x \geq 0 \end{cases}$$

Solution:



The domain of $f(x)$ is $\mathbb{R}$ while the range is $y \in (-\infty, 0] \cup (1, \infty)$.

Question 10 (*12 points*)

Given

$$f(x) = x^2 + 4x + 3 \text{ and } g(x) = x + 3$$

(4 points) (a) Find, algebraically, the points of intersection of $f(x)$ and $g(x)$:

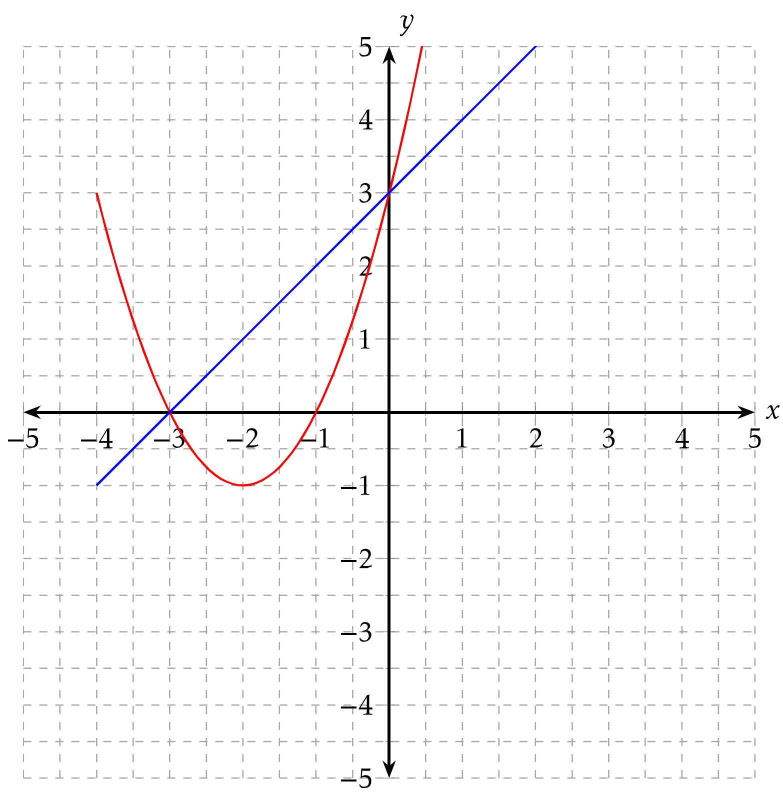
Solution: We want to know when $f(x) = g(x)$:

$$x^2 + 4x + 3 = x + 3$$
$$x^2 + 3x = 0$$
$$x(x + 3) = 0$$

By the Zero Factor Property, $x = -3$ ($y = 0$) or $x = 0$ ($y = 3$).

(8 points) (b) State the range, $x-$ and $y-$ intercepts, and the vertex of f. Graph both functions in the Cartesian Plane provided below:

Solution:
The y-intercept is at $(0, c)$ or $(0, 3)$. Note that $f(x) = (x+1)(x+3)$, so by the Zero Factor Property, -3 or $x = -1$ (the x-intercepts). The x position of the vertex is at $-b/2a$ or $-4/(2 \cdot 1) = -2$, so that $f(-2) = -1$, and the vertex is at $(-2, -1)$. Since the parabola opens upward ($a = 1 > 0$), so the range is $[-1, \infty)$.



Question 11 (*7 points*)

Consider the functions $f(x) = \sqrt{x}$ and $g(x) = \dfrac{x}{x-1}$

(3 points) (a) Find $g^{-1}(x)$.

Solution:

$$\begin{aligned} y &= \frac{x}{x-1} \\ x &= \frac{y}{y-1} \\ x(y-1) &= y \\ xy - x &= y \\ xy - y &= x \\ y(x-1) &= x \\ y &= \frac{x}{x-1} \end{aligned}$$

(3 points) (b) State the domain and the range of $g(x)$.

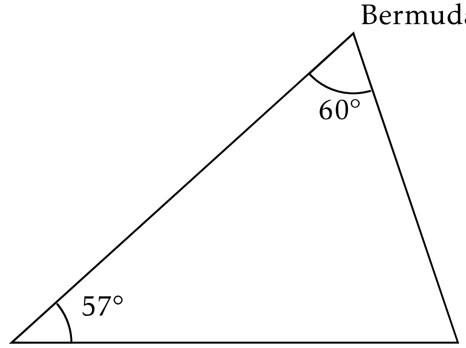
Solution:
The domain of $\mathbb{R}\backslash\{1\}$ since $x-1 \neq 0$. The range of $g(x)$ is the domain of $g^{-1}(x)$ which is the same as that of $g(x)$; hence, the range of $g(x)$ is $\mathbb{R}\backslash\{1\}$.

(1 point) (c) Find $(g \circ g^{-1})(4)$.

Solution:
Since $(g \circ g^{-1})(x) = x$, we have $(g \circ g^{-1})(4) = 4$.

Question 12 (*4 points*)

The Bermuda Triangle is a loosely defined region in the western part of the North Atlantic Ocean. The distance from Bermuda to Miami is 1045 miles (see figure below).



Find the area of the region captured by The Bermuda Triangle.

Solution:

First, note that the angle at Puerto Rico is $180° - 60° - 57° = 63°$. If the distance from Bermuda and Puerto Rico is x, then

$$\frac{\sin(57°)}{x} = \frac{\sin(63°)}{1045}$$

So,

$$x = 1045 \times \frac{\sin(57°)}{\sin(63°)} \approx 984$$

Then, if we drop the height from Puerto Rico until the opposite side:

$$A = \frac{1}{2} \cdot 1045 \cdot 984 \cdot \sin(60°) \approx 445,258 \text{miles}^2$$

Question 13 (*10 points*)

Answer the questions below

(4 points) (a) Show the statement below is true

$$(1-\sin(\theta))(\sec(\theta)+\tan(\theta)) = \cos(\theta)$$

Solution: Let us work on the left side:

$$\begin{aligned}
(1-\sin(\theta))(\sec(\theta)+\tan(\theta)) &= \sec(\theta)+\tan(\theta)-\sin(\theta)\sec(\theta)-\sin(\theta)\tan(\theta) \\
&= \frac{1}{\cos(\theta)}+\frac{\sin(\theta)}{\cos(\theta)}-\sin(\theta)\cdot\frac{1}{\cos(\theta)}-\sin(\theta)\cdot\frac{\sin(\theta)}{\cos(\theta)} \\
&= \frac{1}{\cos(\theta)}-\frac{\sin^2(\theta)}{\cos(\theta)} \\
&= \frac{1-\sin^2(\theta)}{\cos(\theta)} \\
&= \frac{\cos^2(\theta)}{\cos(\theta)} \\
&= \cos(\theta)
\end{aligned}$$

which is the right hand-side.

(3 points) (b) Consider the angle θ in standard position. Find the quadrant that its terminal side lies in , if:

$$\cot\theta > 0,\ \cos\theta < 0$$

Solution: If $\cos\theta < 0$, then the terminal side is in either the second or third quadrant. Since $\cot\theta = \frac{\cos\theta}{\sin\theta} > 0$ and $\cos\theta < 0$, then $\sin\theta < 0$ which means that we are in the third or fourth quadrant.

Overall, we are in the third quadrant.

(3 points) (c) Find the exact value of:

$$\sin\left[\arccos\left(\frac{2}{3}\right)\right]$$

Solution: Let $\theta = \arccos\left(\frac{2}{3}\right)$ so that $\cos\theta = 2/3$. Recall, $\cos\theta = x/r$, so $x = 2$ and $r = 3$ where $r^2 = x^2 + y^2$; hence,

$$y = \sqrt{r^2 - x^2} = \sqrt{9-4} = \sqrt{5}$$

Note that $\theta \in [0, \pi]$, so we are in the first or second quadrant. Since $x = 2 > 0$, we are in the first quadrant, so y is also positive.

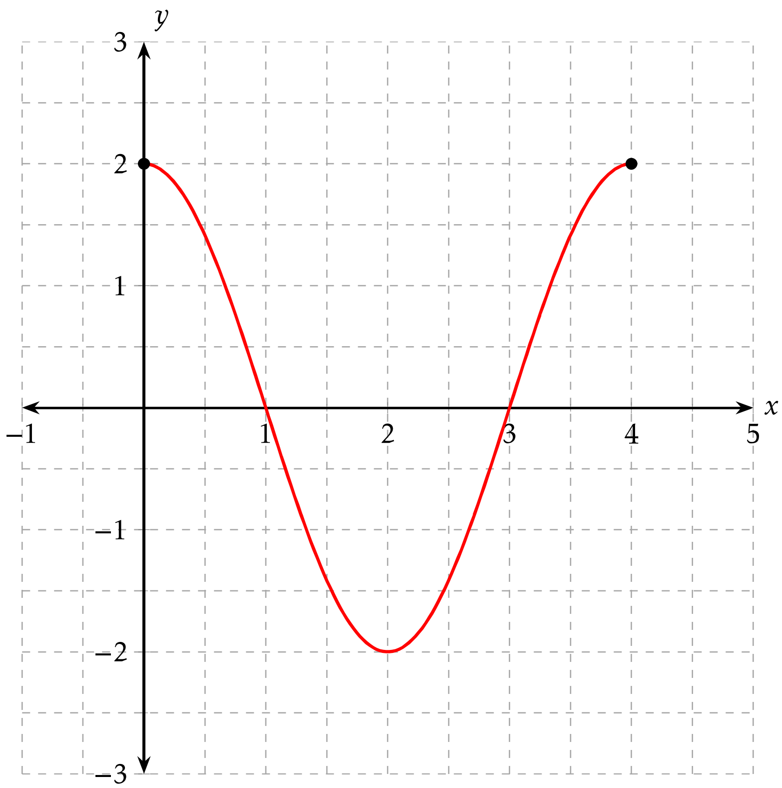
Then, $\sin\theta = y/r = \frac{\sqrt{5}}{3}$.

Question 14 (*3 points*)

Graph $y = 2\cos\left(\frac{x\pi}{2}\right)$ over one period. State the amplitude and period of the function.

Solution: We see that the amplitude of y is $|a| = |2| = 2$ while the period, given $b = \pi/2$, is

$$\frac{2\pi}{b} = \frac{2\pi}{\pi/2} = 4$$



Question 15 (*6 points*)

Solve for x, giving the exact solution where possible, $0 \leq x < 2\pi$.

$$\sin(x) - 2\sin^2(x) = 0$$

Solution:

$$\sin(x) - 2\sin^2(x) = 0$$
$$\sin(x)(1 - 2\sin(x)) = 0$$

By the Zero Factor Property, $\sin(x) = 0$ or $1 - 2\sin(x) = 0$. The former is true for $x = 0,\ \pi$ while the latter implies that $\sin(x) = 1/2$ which is true when $x = \pi/6,\ 5\pi/6$.

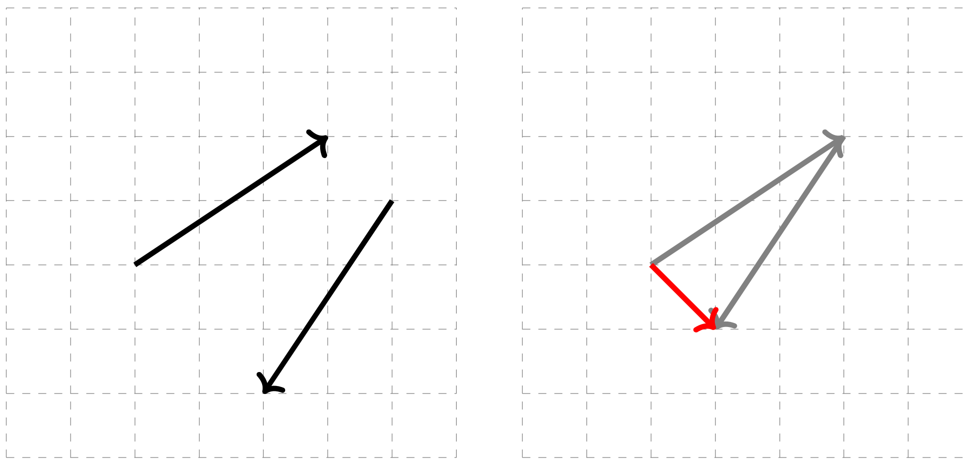
Overall, the solutions are $x = 0,\ \pi/6,\ 5\pi/6,\ \pi$.

Question 16 (*2 points*)

Answer the questions below:

(1 point) (a) Add the given vectors by drawing the appropriate resultant **(please, draw the resultant on the right)**.

Solution:
The resultant is drawn in red while the vectors were placed head-to-tail to emphasis where the resultant is.



(1 point) (b) Are the vectors below perpendicular?

$$\vec{v} = (3, -4) \quad \vec{u} = (8, 6)$$

Solution: Two vectors are perpendicular if the dot product is zero:

$$\vec{v} \cdot \vec{u} = 3 \cdot 8 - 4 \cdot (6) = 24 - 24 = 0$$

Hence, the two vectors are indeed perpendicular.

Thank you for your hard work! It was a pleasure teaching you. Best wishes for the future!