DAWSON COLLEGE

Mathematics Department

Final Examination

201-105-DW (Linear Algebra), Sections 1,2,3,4,6,7,8,

Date: Wednesday, Dec 11, 2019, Time: 9:30 – 12:30

Examiners: O. Cerba, A. Jimenez, S. Shahabi, O. Zlotchevskaia.

Student's Name: ____________________ **ID:** ____________

- Print your full name and student ID number in the space provided above;
- All questions are to be answered directly on the examination paper in the space provided. If you need more space for your answer, use the back of the page;
- No book, notes, or electronic devices are permitted. You are only permitted to use the **Sharp EL-531X**, **Sharp EL-531XG** or **Sharp EL-531XT** calculator;
- Show all your work and justify all your answers;
- This examination booklet must be returned intact;

Solutions.

Question	Marks	student's Score
1	7	
2	12	
3	4	
4	4	
5	4	
6	6	
7	3	
8	8	
9	8	
10	6	
11	9	
12	3	
13	6	
14	6	
15	6	
16	8	
Total	100	

THIS EXAMINATION BOOKLET CONTAINS 11 PAGES (INCLUDING THIS COVER PAGE), AND 16 QUESTIONS.

(1) [5+2 marks] **(i)** Find the general solution of the following system by Gauss (or Gauss-Jordan) Elimination Process:

$$\begin{cases} x_1 - x_2 + x_3 = 9 \\ 2x_1 - 2x_2 + 3x_3 - 2x_4 = 23 \\ 4x_1 - 4x_2 + 3x_3 + 2x_4 = 31 \\ -4x_1 + 4x_2 + 3x_3 - 14x_4 = -1 \end{cases}$$

(ii) Find the particular solution of the system of **(i)** where $x_1 = 1$ and $x_2 = -3$.

(i)

$$\left(\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 9 \\ 2 & -2 & 3 & -2 & 23 \\ 4 & -4 & 3 & 2 & 31 \\ -4 & 4 & 3 & -14 & -1 \end{array}\right) \quad \begin{array}{l} R_2 \to (-2)R_1 + R_2 \\ R_3 \to (-4)R_1 + R_3 \\ R_4 \to (4)R_1 + R_4 \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 9 \\ 0 & 0 & 1 & -2 & 5 \\ 0 & 0 & -1 & 2 & -5 \\ 0 & 0 & 7 & -14 & 35 \end{array}\right) \quad \begin{array}{l} R_1 \to (-1)R_2 + R_1 \\ \\ R_3 \to (1)R_2 + R_3 \\ R_4 \to (7)R_2 + R_4 \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & -1 & 0 & 2 & 4 \\ 0 & 0 & 1 & -2 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$ The reduced echelon form

The system is consistent; we have two pivot variables(x_1, x_3), and hence two free variables(x_2, x_4)

$$\Rightarrow \text{General solution:} \begin{cases} x_1 = r - 2t + 4 \\ x_2 = r \\ x_3 = 2t + 5 \\ x_4 = t \end{cases}$$

(ii)

$x_2 = -3 \to \boxed{r = -3}$

$x_1 = 1 \to (-3) - 2t + 4 = 1 \to \boxed{t = 0}$

$\to (x_1, x_2, x_3, x_4) = (1, -3, 5, 0)$

(2) Let $A = \begin{pmatrix} 2 & 7 & -1 \\ -3 & -11 & 5 \\ 1 & 4 & -3 \end{pmatrix}$.

(i) [5 marks] Find adj(A) (the adjoint of A.)

$$C_{11} = \begin{vmatrix} -11 & 5 \\ 4 & -3 \end{vmatrix} = 13, \quad C_{12} = -\begin{vmatrix} -3 & 5 \\ 1 & -3 \end{vmatrix} = -4, \quad C_{13} = \begin{vmatrix} -3 & -11 \\ 1 & 4 \end{vmatrix} = -1$$

$$C_{21} = -\begin{vmatrix} 7 & -1 \\ 4 & -3 \end{vmatrix} = 17, \quad C_{22} = \begin{vmatrix} 2 & -1 \\ 1 & -3 \end{vmatrix} = -5, \quad C_{23} = -\begin{vmatrix} 2 & 7 \\ 1 & 4 \end{vmatrix} = -1$$

$$C_{31} = \begin{vmatrix} 7 & -1 \\ -11 & 5 \end{vmatrix} = 24, \quad C_{32} = -\begin{vmatrix} 2 & -1 \\ -3 & 5 \end{vmatrix} = -7, \quad C_{33} = \begin{vmatrix} 2 & 7 \\ -3 & -11 \end{vmatrix} = -1$$

$$\Rightarrow \text{adj}(A) = \begin{pmatrix} 13 & -4 & -1 \\ 17 & -5 & -1 \\ 24 & -7 & -1 \end{pmatrix}^T = \begin{pmatrix} 13 & 17 & 24 \\ -4 & -5 & -7 \\ -1 & -1 & -1 \end{pmatrix}$$

(expansion using row $\underline{\text{one}}$)

(ii) [1 marks] Find A^{-1}, using adj(A).

$$|A| = (2)(13) + (7)(-4) + (-1)(-1) = -1$$

$$\Rightarrow A^{-1} = \frac{1}{|A|}\text{adj}(A) = \begin{pmatrix} -13 & -17 & -24 \\ 4 & 5 & 7 \\ 1 & 1 & 1 \end{pmatrix}$$

(One may verify that $A \cdot A^{-1} = I$)

(iii) [2 marks] *Use A^{-1} found in part (ii)* to solve the system: $\begin{cases} 2x + 7y - z = 13 \\ -3x - 11y + 5z = -10 \\ x + 4y - 3z = 0 \end{cases}$

$$AX = \mathbf{b}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 13 \\ -10 \\ 0 \end{pmatrix} \Rightarrow X = A^{-1}\mathbf{b} = \begin{pmatrix} -13 & -17 & -24 \\ 4 & 5 & 7 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 13 \\ -10 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \begin{matrix} \leftarrow x \\ \leftarrow y \\ \leftarrow z \end{matrix}$$

(iv) [4 marks] Use *Cramer's Rule* to verify your answer of part **(iii)** for y **ONLY**.

(arrow technique!)

$$y = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} 2 & 13 & -1 \\ -3 & -10 & 5 \\ 1 & 0 & -3 \end{vmatrix}}{|A|} = \frac{(60) + (0) + (65) - (10) - (0) - (117)}{-1} = \frac{-2}{-1} = \boxed{2} \checkmark$$

(3) [4 marks] Determine the values of p for which the following system has **(i)** no solution, **(ii)** (exactly) one solution, or **(iii)** infinitely many solutions.

$$\begin{cases} -x_1 + 4x_2 - 2x_3 = 1 \\ -2x_1 + 10x_2 + (2p-4)x_3 = 6 \\ 3x_1 - 11x_2 + (p^2+6)x_3 = 5p-1. \end{cases}$$

$$\left(\begin{array}{ccc|c} -1 & 4 & -2 & 1 \\ -2 & 10 & 2p-4 & 6 \\ 3 & -11 & p^2+6 & 5p-1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -4 & 2 & -1 \\ 0 & 2 & 2p & 4 \\ 0 & 1 & p^2 & 5p+2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -4 & 2 & -1 \\ 0 & 1 & p & 2 \\ 0 & 0 & p^2-p & 5p \end{array}\right)$$

$p^2 - p = 0 \to p = 0$ or 1

(i) If $p = 1 \to$ the last row: $\left(\begin{array}{ccc|c} 0 & 0 & 0 & 5 \end{array}\right)$ No solution.

(ii) If $p \neq 0, 1$ $(p^2 - p \neq 0)$ $\to$ by dividing by p^2-p in the last row : $\left(\begin{array}{ccc|c} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \end{array}\right)$ Exactly one solution

(iii) $p = 0 \to \left(\begin{array}{ccc|c} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 0 & 0 \end{array}\right)$ infinitly many solutions.

(4) [4 marks] Find the matrix M if $(5M + 4I)^{-1} = \begin{pmatrix} 7 & 2 \\ 4 & 1 \end{pmatrix}^T = \begin{pmatrix} 7 & 4 \\ 2 & 1 \end{pmatrix}$

$\to 5M + 4I = \begin{pmatrix} 7 & 4 \\ 2 & 1 \end{pmatrix}^{-1} = \frac{1}{-1}\begin{pmatrix} 1 & -4 \\ -2 & 7 \end{pmatrix} = \begin{pmatrix} -1 & 4 \\ 2 & -7 \end{pmatrix}$

$\to 5M = \begin{pmatrix} -1 & 4 \\ 2 & -7 \end{pmatrix} - 4\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -5 & 4 \\ 2 & -11 \end{pmatrix}$

$\to M = \frac{1}{5}\begin{pmatrix} -5 & 4 \\ 2 & -11 \end{pmatrix} = \begin{pmatrix} -1 & 4/5 \\ 2/5 & -11/5 \end{pmatrix}.$

(5) [4 marks] Let $M = \begin{pmatrix} 1 & 2 & 0 \\ -1 & -2 & 1 \\ 0 & 3 & 0 \end{pmatrix}$. Express M^{-1} as a product of elementary matrices.

$$\begin{pmatrix} 1 & 2 & 0 \\ -1 & -2 & 1 \\ 0 & 3 & 0 \end{pmatrix} \xrightarrow{1} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 3 & 0 \end{pmatrix} \xrightarrow{2} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{3} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{4} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

$$I \xrightarrow{1} E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$I \xrightarrow{2} E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$I \xrightarrow{3} E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$I \xrightarrow{4} E_4 = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Since $E_4 E_3 E_2 E_1 M = I$

we have

$$M^{-1} = E_4 E_3 E_2 E_1 .$$

(There are other solutions as well.)

(6) [3+3 marks] Suppose that A, B and C are invertible matrices. **(i)** First simplify, **as much as possible**, *the left-hand side* of the equality: $(2AC^T)^{-1}(CA^T)^T BX = B^{-1}D^T$ $\longrightarrow$

$$\longrightarrow \tfrac{1}{2}(C^T)^{-1}A^{-1}(A^T)^T C^T BX = B^{-1}D^T \longrightarrow \tfrac{1}{2}(C^T)^{-1}\underbrace{A^{-1}A}_{I} C^T BX = B^{-1}D^T \longrightarrow$$

$$\longrightarrow \tfrac{1}{2}\underbrace{(C^T)^{-1}C^T}_{I} BX = B^{-1}D^T \longrightarrow \tfrac{1}{2} BX = B^{-1}D^T$$

(ii) Now solve for X, if $B = \begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 3 & -1 \end{pmatrix}$.

$$B^{-1} = \begin{pmatrix} -7 & 3 \\ 5 & -2 \end{pmatrix}$$

$$\longrightarrow BX = 2B^{-1}D^T \longrightarrow X = 2\, B^{-1} \cdot B^{-1} \cdot D^T$$

$$= 2\begin{pmatrix} -7 & 3 \\ 5 & -2 \end{pmatrix}\begin{pmatrix} -7 & 3 \\ 5 & -2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \end{pmatrix}$$

$$= 2\begin{pmatrix} 64 & -27 \\ -45 & 19 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \end{pmatrix}$$

$$= 2\begin{pmatrix} 37 & 128 & 219 \\ -26 & -90 & -154 \end{pmatrix}$$

$$= \begin{pmatrix} 74 & 256 & 438 \\ -52 & -180 & -308 \end{pmatrix}$$

(7) [3 marks] Suppose that A and B are *skew-symmetric* matrices, that is, suppose $A^T = -A$, and $B^T = -B$. Also assume that $AB = BA$. Now show that AB is symmetric.

$$(AB)^T = B^T A^T = (-B)(-A) = BA = AB$$

$\longrightarrow$ AB is symmetric!

(8) [4+4 marks] Assume that A and B are 4×4 matrices such that $\det(A) = 3$ and $\det(B) = -2$. Now find

(i) $\det\Big((2B)^{-1}(B^3A^{-2})\Big) = \det\left(\frac{1}{2} B^{-1} B^3 A^{-2}\right)$ $\qquad B^{-1}B^3 = B^2$

$$= \left(\tfrac{1}{2}\right)^4 \cdot (\det(B))^2 \cdot (\det(A))^{-2}$$

$$= \frac{1}{16} \cdot (-2)^2 \cdot (3)^{-2} = \frac{1}{36}$$

(ii) $\det\Big((\frac{1}{2}A)^T \operatorname{adj}(A)\Big) = \det\left(\frac{1}{2} A^T \cdot \operatorname{adj}(A)\right) = \left(\frac{1}{2}\right)^4 \det(A) \det(\operatorname{adj}(A))$

$$= \frac{1}{16} \cdot 3 \cdot (3)^{4-1} = \frac{81}{16}$$

$\Big(\det(\operatorname{adj}(A)) = |A|^{n-1} \Big)$ n: size

(9) [4+4 marks] (i) Evaluate $\underbrace{\begin{vmatrix} a-5g & -a+2d & -3g \\ b-5h & -b+2e & -3h \\ c-5i & -c+2f & -3i \end{vmatrix}}_{D}$, if we know $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -4$.

$$D = (-3) \cdot \begin{vmatrix} a-5g & -a+2d & g \\ b-5h & -b+2e & h \\ c-5i & -c+2f & i \end{vmatrix} = (-3) \cdot \begin{vmatrix} a & -a+2d & g \\ b & -b+2e & h \\ c & -c+2f & i \end{vmatrix}$$

$$= -3 \cdot \begin{vmatrix} a & 2d & g \\ b & 2e & h \\ c & 2f & i \end{vmatrix} = (-3)(2) \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix}$$

$$= -6 \cdot \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = (-6)(-4) = 24.$$

(ii) Evaluate $\overbrace{\begin{vmatrix} 6 & 3 & 2 & -2 \\ 2 & -3 & -1 & 8 \\ 7 & 3 & 4 & -5 \\ -5 & 3 & 1 & -6 \end{vmatrix}}^{D}$ by a *combination of row operations and cofactor expansion.*

No marks will be given otherwise!

$$D = 3 \cdot \begin{vmatrix} 6 & 1 & 2 & -2 \\ 2 & -1 & -1 & 8 \\ 7 & 1 & 4 & -5 \\ -5 & 1 & 1 & -6 \end{vmatrix} = 3 \cdot \begin{vmatrix} 8 & 0 & 1 & 6 \\ 2 & -1 & -1 & 8 \\ 9 & 0 & 3 & 3 \\ -3 & 0 & 0 & 2 \end{vmatrix}$$

$$= 3 \cdot (-1) \begin{vmatrix} 8 & 1 & 6 \\ 9 & 3 & 3 \\ -3 & 0 & 2 \end{vmatrix} = (-9) \cdot \begin{vmatrix} 8 & 1 & 6 \\ 3 & 1 & 1 \\ -3 & 0 & 2 \end{vmatrix}$$

$$= -9 \cdot \begin{vmatrix} 5 & 0 & 5 \\ 3 & 1 & 1 \\ -3 & 0 & 2 \end{vmatrix} = (-9) \cdot (1) \cdot \begin{vmatrix} 5 & 5 \\ -3 & 2 \end{vmatrix} = -9(10+15)$$

$$= -225.$$

(10) [3+3 marks] Let $\vec{u}$ and $\vec{v}$ be two vectors in $\mathbb{R}^3$. Now answer the following:

(i) If $\vec{u} \perp \vec{v}$, then what is $\mathrm{Proj}_{\vec{v}}\vec{u}$? **(Justify your answer!)**

$$\vec{u} \perp \vec{v} \Rightarrow \vec{u} \cdot \vec{v} = 0 \Rightarrow \mathrm{Proj}_{\vec{v}}\vec{u} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \vec{v} = 0 \cdot \vec{v} = \vec{0}$$

the zero vector!

(ii) Simplify the vector expression $(2\vec{u} + 3\vec{v}) \times (4\vec{u} - 5\vec{v})$.

The expression $= 2\vec{u} \times 4\vec{u} - 2\vec{u} \times 5\vec{v} + 3\vec{v} \times 4\vec{u} - 3\vec{v} \times 5\vec{v}$

$= 8 \cdot \vec{u} \times \vec{u} - 10 \cdot \vec{u} \times \vec{v} + 12 \cdot \vec{v} \times \vec{u} - 15 \cdot \vec{v} \times \vec{v}$

$= 8\vec{0} - 10\,\vec{u} \times \vec{v} - 12\,\vec{u} \times \vec{v} - 15\,\vec{0}$

$= -22 \cdot \vec{u} \times \vec{v}$ (or $22 \cdot \vec{v} \times \vec{u}$)

(11) [3+3+3 marks] Let $A(1,1,1)$, $B(-1,0,2)$ and $C(3,2,1)$ be given.

(i) Find a vector of norm 4 which is *oppositely directed* to $\overrightarrow{AB}$.

$\overrightarrow{AB} = (-2,-1,1)$

$\|\overrightarrow{AB}\| = \sqrt{6}$

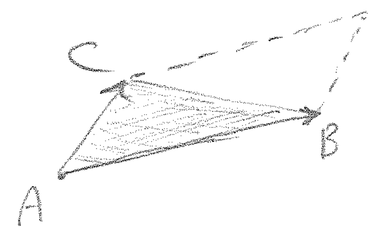
$$\vec{u} = -\frac{4}{\|\overrightarrow{AB}\|} \cdot \overrightarrow{AB} = -\frac{4}{\sqrt{6}}(-2,-1,1)$$
$$= \left(\frac{8}{\sqrt{6}}, \frac{4}{\sqrt{6}}, -\frac{4}{\sqrt{6}}\right)$$

(ii) Find the area of the triangle $\Delta(ABC)$.

$\overrightarrow{AC} = (2,1,0)$

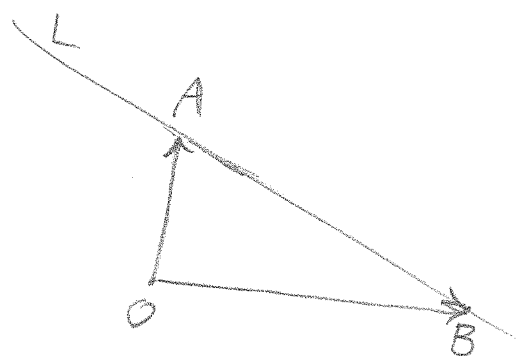
$\overrightarrow{AB} \times \overrightarrow{AC} = (-1,2,0)$

$$\text{area of } \Delta(ABC) = \frac{1}{2}\|\overrightarrow{AB} \times \overrightarrow{AC}\|$$
$$= \frac{1}{2}\|(-1,2,0)\|$$
$$= \frac{1}{2}\sqrt{5}$$



(iii) Find the distance from the origin to the line L passing through A and B.

$$\text{The distance} = \frac{\|\overrightarrow{OA} \times \overrightarrow{OB}\|}{\|\overrightarrow{AB}\|} = \frac{\sqrt{14}}{\sqrt{6}}$$
$$= \sqrt{\frac{7}{3}}$$



$\overrightarrow{OA} = (1,1,1)$

$\overrightarrow{OB} = (-1,0,2)$

$\overrightarrow{AB} = (-2,-1,1)$

$\overrightarrow{OA} \times \overrightarrow{OB} = (2,-3,1)$

(12) [3 marks] Find an equation of the plane that passes through $A(1,2,3)$ and is perpendicular to the line $L:\ x = 2t-3,\ y = -t,\ z = 3t+1$.

Direction vector of the line, namely $\vec{v} = (2,-1,3)$, may be used also as a normal of the desired plane; Hence the equation Π: $2(x-1) - 1(y-2) + 3(z-3) = 0$

or $2x - y + 3z = 9$

(13) [3+3 marks] Consider the plane $\Pi : 3x - 2y + 4z = 16$ and the line $L:\ x = t,\ y = t+1,\ z = t+2$.

(i) Find the point of intersection of the plane Π and the line L.

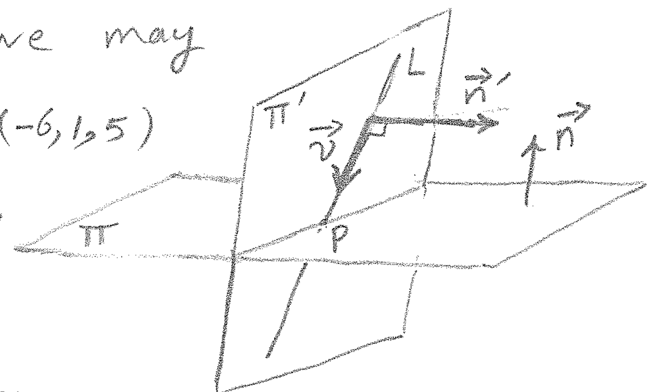
Need to solve: $3(t) - 2(t+1) + 4(t+2) = 16$

$\rightarrow 5t = 10 \rightarrow t = 2 \rightarrow P(2, 3, 4)$

(ii) Find the plane Π' which contains L and which is perpendicular to Π.

Since $\Pi' \perp \Pi \rightarrow \vec{n}' \perp n$

And since Π' contains L, $\vec{n}' \perp \vec{v}$, where $\vec{v}$ is a direction vector of L. Hence we may set $\vec{n}' = \vec{n} \times \vec{v} = (3, -2, 4) \times (1, 1, 1) = (-6, 1, 5)$



Hence: $\Pi' : -6(x-2) + 1\cdot(y-3) + 5(z-4) = 0$

or $-6x + y + 5z - 11 = 0$

(14) [3+3 marks] Consider the planes $\Pi_1 : x - 2y + 7z = 8$ and $\Pi_2 : 2x - 3y + 4z = -3$.

(i) Find the parametric equations of the line of intersection L of Π_1 and Π_2.

- Need to solve

$$\begin{cases} x - 2y + 7z = 8 \\ 2x - 3y + 4z = -3 \end{cases} \quad \left(\begin{array}{ccc|c} 1 & -2 & 7 & 8 \\ 2 & -3 & 4 & -3 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & -2 & 7 & 8 \\ 0 & 1 & -10 & -19 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -13 & -30 \\ 0 & 1 & -10 & -19 \end{array}\right) \qquad \begin{cases} x = 13t - 30 \\ y = 10t - 19 \\ z = t \end{cases}$$

Thus the line of intersection

$L: (x, y, z) = t(13, 10, 1) + (-30, -19, 0)$, or

- 2nd approach: Take $\vec{v} = \vec{n} \times \vec{n}' = (1, -2, 7) \times (2, -3, 4) = (13, 10, 1)$ as a direction vector for the line of intersection.

Now we just need also a point on the line.

Setting $z = 0$, say, leads to solving $\begin{cases} x - 2y = 8 \\ 2x - 3y = -3 \end{cases} \rightarrow x = -30, y = -19$

Thus the (same) answer: $L \begin{cases} x = 13t - 30 \\ y = 10t - 19 \\ z = t \end{cases}$

(ii) (continuation of Problem (14)) Find the distance between the point $P(-1, 2, -4)$ and the plane Π_1.

$\Pi_1: \ x - 2y + 7z - 8 = 0$

$$\text{the distance} = \frac{|(-1) - 2(2) + 7(-4) - 8|}{\sqrt{1^2 + (-2)^2 + 7^2}} = \frac{|-41|}{\sqrt{54}} = \frac{41}{\sqrt{54}}$$

(15) [3+3 marks] Consider the lines $L_1 : \begin{cases} x = 3s - 2 \\ y = s \\ z = -4s + 5 \end{cases}$ and $L_2 : \begin{cases} x = 4t + 1 \\ y = -3t + 1 \\ z = t + 1. \end{cases}$

(i) Find the point of intersection of L_1 and L_2.

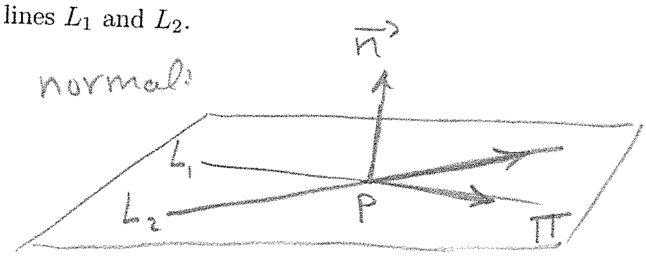
Need to solve $\begin{cases} 3s - 2 = 4t + 1 \\ s = -3t + 1 \\ -4s + 5 = t + 1 \end{cases}$ or $\begin{cases} 3s - 4t = 3 \\ s + 3t = 1 \\ -4s - t = -4 \end{cases}$

$$\left(\begin{array}{cc|c} 3 & -4 & 3 \\ 1 & 3 & 1 \\ -4 & -1 & -4 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 3 & 1 \\ 0 & -13 & 0 \\ 0 & 11 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 3 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \longrightarrow t = 0, \ s = 1$$

$\longrightarrow P(1, 1, 1)$

(ii) Find an equation of the plane which contains both lines L_1 and L_2.

We may use $\vec{n} = \vec{v_1} \times \vec{v_2}$ as a normal:



$\vec{n} = (3, 1, -4) \times (4, -3, 1)$

$= (-11, -19, -13)$

Using the point P from (i), we thus get

$\Pi: \ -11(x-1) - 19(y-1) - 13(z-1) = 0$

or $11x + 19y + 13z = 43$

(16) [8 marks] Maximize $P = 6x_1 + 2x_2 + 8x_3$, subject to the constraints $\begin{cases} x_1 + 3x_2 + x_3 \leq 9 \\ x_1 + 6x_2 - x_3 \leq 2 \\ 2x_2 + x_3 \leq 5 \\ (x_1, x_2, x_3 \geq 0). \end{cases}$

The simplex "Tableau":

$$\left(\begin{array}{ccc|ccc|c|c} 1 & 3 & 1 & 1 & 0 & 0 & 0 & 9 \\ 1 & 6 & -1 & 0 & 1 & 0 & 0 & 2 \\ 0 & 2 & 1 & 0 & 0 & 1 & 0 & 5 \\ \hline -6 & -2 & -8 & 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccccccc|c} 1 & 1 & 0 & 1 & 0 & -1 & 0 & 4 \\ 1 & 8 & 0 & 0 & 1 & 1 & 0 & 7 \\ 0 & 2 & 1 & 0 & 0 & 1 & 0 & 5 \\ \hline -6 & 14 & 0 & 0 & 0 & 8 & 1 & 40 \end{array}\right)$$

$$\left(\begin{array}{ccccccc|c} 1 & 1 & 0 & 1 & 0 & -1 & 0 & 4 \\ 0 & 7 & 0 & -1 & 1 & 2 & 0 & 3 \\ 0 & 2 & 1 & 0 & 0 & 1 & 0 & 5 \\ \hline 0 & 20 & 0 & 6 & 0 & 2 & 1 & 64 \end{array}\right)$$

x_1, x_3: basic variables $\quad x_2$: non-basic

$\Rightarrow$ the solution $\begin{cases} x_1 = 4 \\ x_2 = 0 \\ x_3 = 5 \end{cases}$.

Max $(P) = 64$