DAWSON COLLEGE
Mathematics Department
Final Examination
Linear Algebra
201-105 –DW
May 20, 2022

Student Name Solutions

Student I.D. # ______________________

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TIME: 2:00 p.m. – 5:00 p.m.

Instructions:

- Print your name and student ID number in the space provided above.
- All questions are to be answered directly on the examination paper in the provided space.
- Translation and regular dictionaries are permitted.
- The Sharp EL-531** calculator are permitted.
- This examination consists of 15 questions.
- There are 12 pages including this cover page.
- **This exam booklet must be returned intact.**

Question #	Marks
1/7	
2/9	
3/10	
4/4	
5/4	
6/4	
7/4	
8/12	
9/11	
10/7	
11/3	
12/11	
13/3	
14/3	
15/8	
Total / 100	

25% Class Marks = ______
+
75% Final Exam = ______
OR
50% Class Marks = ______
+
50% Final Exam = ______
Total = ______

FINAL GRADE = ______

1. a) (5 marks) Use Gauss-Jordan elimination to find the general solution of the system.
   b) (2 mark) Find a particular solution of the system in which $x_2 = 1$.

$$\begin{cases} x_1 - 2x_2 - 2x_3 + 3x_4 = 7 \\ 3x_1 - 5x_2 + x_3 + 8x_4 = 13 \\ -x_1 + 2x_2 + 2x_3 - 4x_4 = -9 \end{cases}$$

a)

$$\left[\begin{array}{cccc|c} 1 & -2 & -2 & 3 & 7 \\ 3 & -5 & 1 & 8 & 13 \\ -1 & 2 & 2 & -4 & -9 \end{array}\right] \begin{array}{l} R_1\cdot(-3)+R_2 \to R_2 \\ R_1 + R_3 \to R_3 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -2 & -2 & 3 & 7 \\ 0 & 1 & 7 & -1 & -8 \\ 0 & 0 & 0 & -1 & -2 \end{array}\right] R_3\cdot(-1)$$

$$\left[\begin{array}{cccc|c} 1 & -2 & -2 & 3 & 7 \\ 0 & 1 & 7 & -1 & -8 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} R_3\cdot(-3)+R_1 \to R_1 \\ R_3 + R_2 \to R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -2 & -2 & 0 & 1 \\ 0 & 1 & 7 & 0 & -6 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] R_2\cdot 2 + R_1 \to R_1$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 12 & 0 & -11 \\ 0 & 1 & 7 & 0 & -6 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

$$\begin{cases} x_1 + 12x_3 = -11 \\ x_2 + 7x_3 = -6 \\ x_4 = 2 \end{cases}$$

$$\begin{cases} x_1 = -11 - 12x_3 \\ x_2 = -6 - 7x_3 \\ x_4 = 2 \end{cases}$$

The general solution is

$$\begin{cases} x_1 = -11 - 12t \\ x_2 = -6 - 7t \\ x_3 = t \\ x_4 = 2 \end{cases}, \quad t \in R.$$

b) $x_2 = 1$

$-6 - 7t = 1$

$-7t = 7$

$t = -1$

$x_1 = -11 - 12\cdot(-1) = 1$

$x_3 = -1$

Ans. $(1, 1, -1, 2)$

2. (4+5 marks) Given the following matrices $A=\begin{bmatrix}3 & 7\\ 1 & 2\end{bmatrix}$, $B=\begin{bmatrix}2 & -1\\ -1 & 0\end{bmatrix}$ and $C=\begin{bmatrix}4 & -1\\ 0 & -3\\ 1 & 2\end{bmatrix}$.

a) Calculate $tr\left(2C^TC+3B^2-I^{2022}\right)$

$$C^TC=\begin{bmatrix}4 & 0 & 1\\ -1 & -3 & 2\end{bmatrix}\cdot\begin{bmatrix}4 & -1\\ 0 & -3\\ 1 & 2\end{bmatrix}=\begin{bmatrix}17 & -2\\ -2 & 14\end{bmatrix},\quad 2C^TC=\begin{bmatrix}34 & -4\\ -4 & 28\end{bmatrix}$$

$$B^2=\begin{bmatrix}2 & -1\\ -1 & 0\end{bmatrix}\cdot\begin{bmatrix}2 & -1\\ -1 & 0\end{bmatrix}=\begin{bmatrix}5 & -2\\ -2 & 1\end{bmatrix},\quad I^{2022}=I$$

$$tr\left(2C^TC+3B^2-I^{2022}\right)=tr\left(\begin{bmatrix}34 & -4\\ -4 & 28\end{bmatrix}+\begin{bmatrix}15 & -6\\ -6 & 3\end{bmatrix}-\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)$$

$$=tr\left(\begin{bmatrix}48 & -10\\ -10 & 30\end{bmatrix}\right)=48+30=78$$

b) Solve for X: $\left(BX^T-3A\right)^{-1}=A$,

$$BX^T-3A=A^{-1}$$

$$BX^T=A^{-1}+3A$$

$$BX^T=\begin{bmatrix}-2 & 7\\ 1 & -3\end{bmatrix}+\begin{bmatrix}9 & 21\\ 3 & 6\end{bmatrix}$$

$$BX^T=\begin{bmatrix}7 & 28\\ 4 & 3\end{bmatrix}$$

$$A^{-1}=\begin{bmatrix}3 & 7\\ 1 & 2\end{bmatrix}^{-1}=\frac{1}{6-7}\begin{bmatrix}2 & -7\\ -1 & 3\end{bmatrix}=\begin{bmatrix}-2 & 7\\ 1 & -3\end{bmatrix}$$

$$B^{-1}=\begin{bmatrix}2 & -1\\ -1 & 0\end{bmatrix}=\frac{1}{0-1}\begin{bmatrix}0 & 1\\ 1 & 2\end{bmatrix}=\begin{bmatrix}0 & -1\\ -1 & -2\end{bmatrix}$$

$$X^T=B^{-1}\begin{bmatrix}7 & 28\\ 4 & 3\end{bmatrix}$$

$$X^T=\begin{bmatrix}0 & -1\\ -1 & -2\end{bmatrix}\cdot\begin{bmatrix}7 & 28\\ 4 & 3\end{bmatrix}$$

$$X^T=\begin{bmatrix}-4 & -3\\ -15 & -34\end{bmatrix}$$

$$\boxed{X=\begin{bmatrix}-4 & -15\\ -3 & -34\end{bmatrix}}$$

3. (5+2+3 marks) Given the system of linear equations $\begin{cases} x+3y+z=13 \\ x-y-2z=-7 \\ 2x+y-z=4 \end{cases}$.

a) Use the adjoint matrix to find the inverse of the coefficient matrix.

$$\det(A) = \begin{vmatrix} 1 & 3 & 1 \\ 1 & -1 & -2 \\ 2 & 1 & -1 \end{vmatrix} = 1\cdot\begin{vmatrix} -1 & -2 \\ 1 & -1 \end{vmatrix} - 3\begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix} + 1\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = 1\cdot 3 - 3\cdot 3 + 1\cdot 3 = \boxed{-3}$$

$$C_{11} = \begin{vmatrix} -1 & -2 \\ 1 & -1 \end{vmatrix} = 3, \quad C_{12} = -\begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix} = -3, \quad C_{13} = \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = 3$$

$$C_{21} = -\begin{vmatrix} 3 & 1 \\ 1 & -1 \end{vmatrix} = 4, \quad C_{22} = \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -3, \quad C_{23} = -\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = 5$$

$$C_{31} = \begin{vmatrix} 3 & 1 \\ -1 & -2 \end{vmatrix} = -5, \quad C_{32} = -\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = 3, \quad C_{33} = \begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix} = -4$$

$$\operatorname{adj}(A) = \begin{bmatrix} 3 & -3 & 3 \\ 4 & -3 & 5 \\ -5 & 3 & -4 \end{bmatrix}^T = \begin{bmatrix} 3 & 4 & -5 \\ -3 & -3 & 3 \\ 3 & 5 & -4 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) = \frac{1}{-3}\begin{bmatrix} 3 & 4 & -5 \\ -3 & -3 & 3 \\ 3 & 5 & -4 \end{bmatrix} = \begin{bmatrix} -1 & -\frac{4}{3} & \frac{5}{3} \\ 1 & 1 & -1 \\ -1 & -\frac{5}{3} & \frac{4}{3} \end{bmatrix}$$

b) Use the inverse of the coefficient matrix to solve the system.

$$X = A^{-1}B = \begin{bmatrix} -1 & -\frac{4}{3} & \frac{5}{3} \\ 1 & 1 & -1 \\ -1 & -\frac{5}{3} & \frac{4}{3} \end{bmatrix} \cdot \begin{bmatrix} 13 \\ -7 \\ 4 \end{bmatrix} = \begin{bmatrix} -13 + \frac{28}{3} + \frac{20}{3} \\ 13 - 7 - 4 \\ -13 + \frac{35}{3} + \frac{16}{3} \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$$

Ans. $\boxed{x = 3,\ y = 2,\ z = 4.}$

c) Use Cramer's rule to solve the system for y only.

$$\det(A_2) = \begin{vmatrix} 1 & 13 & 1 \\ 1 & -7 & -2 \\ 2 & 4 & -1 \end{vmatrix} = 1\cdot\begin{vmatrix} -7 & -2 \\ 4 & -1 \end{vmatrix} - 13\begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix} + 1\cdot\begin{vmatrix} 1 & -7 \\ 2 & 4 \end{vmatrix}$$

$$= 1\cdot 15 - 13\cdot 3 + 1\cdot 18 = -6$$

$$y = \frac{\det(A_2)}{\det(A)} = \frac{-6}{-3} = 2.$$

4. (4 marks) If A, B and C are $n\times n$ invertible matrices then simplify the following expression

$$\left(2BC^T\right)^{-1}\cdot\left(B^{-1}CB^T\right)^T\cdot\left(8A^0B\right)^T.$$

$$=\frac{1}{2}\left(C^T\right)^{-1}B^{-1}\cdot\left(B^T\right)^TC^T\left(B^{-1}\right)^T\cdot 8B^T\left(A^0\right)^T$$

$$=\frac{8}{2}\left(C^T\right)^{-1}\underbrace{B^{-1}\cdot B}_{I}\cdot C^T\underbrace{\left(B^T\right)^{-1}\cdot B^T}_{I}\cdot I^T$$

$$=4\left(C^T\right)^{-1}\cdot I\,C^T\,I\cdot I$$

$$=4\left(C^T\right)^{-1}C^T=4I$$

5. (4 marks) For which values of k does the following system $\begin{cases} x+5y-3z=2 \\ -2x-7y+3z=-5 \\ -x-5y+\left(k^2-6\right)z=k+1 \end{cases}$ have

1) exactly one solution, 2) infinitely many solutions, 3) no solution?

$$\left[\begin{array}{ccc|c} 1 & 5 & -3 & 2 \\ -2 & -7 & 3 & -5 \\ -1 & -5 & k^2-6 & k+1 \end{array}\right] \quad \begin{array}{l} R_1\cdot 2+R_2\to R_2 \\ R_1+R_3\to R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 5 & -3 & 2 \\ 0 & 3 & -3 & -1 \\ 0 & 0 & k^2-9 & k+3 \end{array}\right]$$

1) If $k^2-9\neq 0$, $k\neq\pm 3$

then $R_3\cdot\frac{1}{k^2-9}$, $R_2\cdot\frac{1}{3}$

$$\left[\begin{array}{ccc|c} 1 & 5 & -3 & 2 \\ 0 & 1 & -1 & -\frac{1}{3} \\ 0 & 0 & 1 & \frac{k+3}{k^2-9} \end{array}\right]$$

Exactly one solution

2) If $k^2-9=0$, then

$k=3$ or $k=-3$

$$\left[\begin{array}{ccc|c} 1 & 5 & -3 & 2 \\ 0 & 3 & -3 & -1 \\ 0 & 0 & 0 & 6 \end{array}\right]$$

No solutions

$$\left[\begin{array}{ccc|c} 1 & 5 & -3 & 2 \\ 0 & 3 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Inf. many solutions

$$\boxed{\begin{array}{l} \text{Ans. } 1)\ k\neq\pm 3 \\ 2)\ k=-3 \\ 3)\ k=3 \end{array}}$$

6. (4 marks) Evaluate the determinant by a combination of row operations and cofactor expansion. **You must perform at least one row operation.**

$$\begin{vmatrix} -2 & 2 & -4 & -1 \\ 5 & 0 & -1 & -3 \\ 1 & -2 & 6 & 2 \\ 3 & -2 & 4 & 3 \end{vmatrix} \quad R_1 + R_3 \to R_3 \quad R_1 + R_4 \to R_4$$

$$= \begin{vmatrix} -2 & 2 & -4 & -1 \\ 5 & 0 & -1 & -3 \\ -1 & 0 & 2 & 1 \\ 1 & 0 & 0 & 2 \end{vmatrix} = -2 \begin{vmatrix} 5 & -1 & -3 \\ -1 & 2 & 1 \\ 1 & 0 & 2 \end{vmatrix} \quad R_1 \cdot 2 + R_2 \to R_2$$

$$= -2 \cdot \begin{vmatrix} 5 & -1 & -3 \\ 9 & 0 & -5 \\ 1 & 0 & 2 \end{vmatrix} = -2 \cdot 1 \cdot \begin{vmatrix} 9 & -5 \\ 1 & 2 \end{vmatrix} = -2(18+5)$$

$$= -46$$

7. (4 marks) If $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 5$, find $\begin{vmatrix} a+3g & -g & -5a+d \\ b+3h & -h & -5b+e \\ c+3i & -i & -5c+f \end{vmatrix}$.

col.2 · 3 + col.1 → col.1

$$= \begin{vmatrix} a & -g & -5a+d \\ b & -h & -5b+e \\ c & -i & -5c+f \end{vmatrix} \quad \text{col.1} \cdot 5 + \text{col.3} \to \text{col.3}$$

$$= \begin{vmatrix} a & -g & d \\ b & -h & e \\ c & -i & f \end{vmatrix} = -\begin{vmatrix} a & g & d \\ b & h & e \\ c & i & f \end{vmatrix} \quad \text{transpose}$$

$$= -\begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix} \quad R_2 \leftrightarrow R_3 = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 5$$

8. (12 marks) Let A, B, C and D be 2×2 matrices and $\det(A)=-2$, $\det(B)=3$.

a) Find $\det\left(-5A^2\cdot(3B)^{-1}\det(B)\right)$

$= \det(-5A^2\cdot(3B)^{-1}\cdot 3)$

$= \det(-15A^2\cdot(3B)^{-1})$

$= (-15)^2(\det(A))^2\cdot\dfrac{1}{\det(3B)}$

$= 225\cdot(-2)^2\cdot\dfrac{1}{3^2\det(B)}$

$= \dfrac{225\cdot 4}{3^2\cdot 3} = \dfrac{100}{3}$

b) Find $\det\left(6A^{-1}+adj(A)\right)$

$= \det(6A^{-1}+A^{-1}\det(A))$

$= \det(6A^{-1}+A^{-1}\cdot(-2))$

$= \det(6A^{-1}-2A^{-1})$

$= \det(4A^{-1}) = 4^2\cdot\dfrac{1}{\det(A)} = 16\cdot\dfrac{1}{-2} = -8$

c) Find $\det(C)$, if $\det\left(4D^{-1}B^{-2}C^2D\right)=16$

$4^2\dfrac{1}{\cancel{\det(D)}}\cdot\dfrac{1}{(\det(B))^2}\cdot(\det(C))^2\cdot\cancel{\det(D)}=16$

$16\cdot\dfrac{1}{3^2}(\det(C))^2=16$

$(\det(C))^2=9$

$\boxed{\det(C)=\pm 3}$

9. (3+4+4 marks) Consider points $A(2,-1,3)$, $B(1,-3,4)$ and $C(3,-1,2)$.

a) Find a unit vector in the same direction as $\overrightarrow{AB}$.

$$\overrightarrow{AB} = (-1,-2,1)$$

$$\frac{1}{\|\overrightarrow{AB}\|}\overrightarrow{AB} = \frac{1}{\sqrt{(-1)^2+(-2)^2+1^2}}(-1,-2,1) = \frac{1}{\sqrt{6}}(-1,-2,1)$$

$$= \boxed{\left(-\frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)}$$

b) Find the area of the triangle with vertices A, B and C.

$$\overrightarrow{AC} = (1,0,-1)$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & -2 & 1 \\ 1 & 0 & -1 \end{vmatrix} = (2,0,2)$$

$$\text{Area}_{\Delta ABC} = \frac{1}{2}\|\overrightarrow{AB}\times\overrightarrow{AC}\| = \frac{1}{2}\sqrt{2^2+0^2+2^2} = \frac{1}{2}\sqrt{8} \overset{\text{OR}}{=} \sqrt{2}$$

c) Find the angle (in degrees) at the vertex A of the triangle ABC. Approximate the answer to two decimal places.

$$\cos\angle A = \frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{\|\overrightarrow{AB}\|\cdot\|\overrightarrow{AC}\|} = \frac{-1\cdot 1 - 2\cdot 0 + 1\cdot(-1)}{\sqrt{6}\cdot\sqrt{1^2+0^2+(-1)^2}} = \frac{-2}{\sqrt{6}\sqrt{2}} \overset{\text{OR}}{=} -\frac{1}{\sqrt{3}}$$

$$\angle A = \cos^{-1}\left(\frac{-2}{\sqrt{6}\sqrt{2}}\right) \approx 125.26^\circ$$

10. (4+3 marks) Let $\vec{u}=(2,-1,\ 3)$, $\vec{v}=(1,-2,-4)$.

a) Find the orthogonal projection of the vector $\vec{u}+\vec{v}$ on the vector $\vec{u}-2\vec{v}$, that is $Proj_{\ \vec{u}-2\vec{v}}\,(\vec{u}+\vec{v})$.

$$\vec{a}=\vec{u}-2\vec{v}=(2,-1,3)-(2,-4,-8)=(0,3,11)$$

$$\vec{u}+\vec{v}=(3,-3,-1)$$

$$\text{proj}_{\vec{u}-2\vec{v}}(\vec{u}+\vec{v})=\text{proj}_{\vec{a}}(\vec{u}+\vec{v})=\frac{(\vec{u}+\vec{v})\cdot\vec{a}}{\|\vec{a}\|^2}\vec{a}$$

$$=\frac{3\cdot 0-3\cdot 3-1\cdot 11}{0^2+3^2+11^2}\vec{a}=\frac{-20}{130}\vec{a}=\frac{-2}{13}(0,3,11)=\boxed{\left(0,-\frac{6}{13},-\frac{22}{13}\right)}$$

b) Find the volume of the parallelepiped determined by $\vec{u}+\vec{v}$, $\vec{v}$ and $\vec{w}=(2,-3,-4)$.

$$(\vec{u}+\vec{v})\cdot(\vec{v}\times\vec{w})=\begin{vmatrix} 3 & -3 & -1 \\ 1 & -2 & -4 \\ 2 & -3 & -4 \end{vmatrix}=3\begin{vmatrix} -2 & -4 \\ -3 & -4 \end{vmatrix}+3\begin{vmatrix} 1 & -4 \\ 2 & -4 \end{vmatrix}-1\cdot\begin{vmatrix} 1 & -2 \\ 2 & -3 \end{vmatrix}$$

$$=3(8-12)+3(-4+8)-1\cdot(-3+4)=-12+12-1=-1$$

$$\text{Volume}=|(\vec{u}+\vec{v})\cdot(\vec{v}\times\vec{w})|=|-1|=1$$

11. (3 marks) Suppose $\vec{a}\cdot(\vec{b}\times\vec{c})=5$. Find $(\vec{a}-\vec{b})\cdot(\vec{a}\times\vec{c})$.

$$(\vec{a}-\vec{b})\cdot(\vec{a}\times\vec{c})$$

$$=\vec{a}\cdot(\vec{a}\times\vec{c})-\vec{b}\cdot(\vec{a}\times\vec{c})$$

$$=\underbrace{\begin{vmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}}_{=0}-\underset{R_1\leftrightarrow R_2}{\begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}}=+\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}=\vec{a}\cdot(\vec{b}\times\vec{c})=5$$

(two identical rows)

12. (4+3+4 marks) Given the point $A(1,\ 4,-1)$, the plane P: $x-3y+2z=1$ and the line L: $\begin{cases} x=3+t \\ y=6-t \\ z=1+2t \end{cases}$.

a) Find the equation of the plane that contains point A and is perpendicular to the line L.

$$\vec{n}=\vec{v}=(1,-1,2)$$

$$1\cdot(x-1)-1\cdot(y-4)+2(z+1)=0$$

$$x-1-y+4+2z+2=0$$

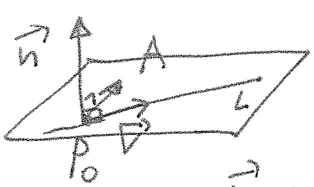
$$\boxed{x-y+2z+5=0}$$

b) Find the distance from the point A to the plane P. $x-3y+2z-1=0$

$$D=\frac{|1-3\cdot4+2\cdot(-1)-1|}{\sqrt{1^2+(-3)^2+2^2}}=\frac{|-14|}{\sqrt{14}}=\frac{14}{\sqrt{14}} \overset{\text{OR}}{=} \sqrt{14}$$

c) Find the equation of the plane containing the point A and the line L. $P_0(3,6,1)$

$\vec{v}=(1,-1,2)$



$$\overrightarrow{P_0A}=(-2,-2,-2)$$

$$\vec{n}=\overrightarrow{P_0A}\times\vec{v}=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & -2 & -2 \\ 1 & -1 & 2 \end{vmatrix}=(-6,2,4)$$

$$-6(x-1)+2(y-4)+4(z+1)=0$$

$$-6x+6+2y-8+4z+4=0$$

$$\boxed{-6x+2y+4z+2=0}$$

OR $\boxed{3x-y-2z-1=0}$

13. (3 marks) Find the parametric equations of the line of intersection of the planes $x-3y+2z=1$ and $3x-8y+4z=8$.

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & 1 \\ 3 & -8 & 4 & 8 \end{array}\right] R_1\cdot(-3)+R_2\to R_2$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & 1 \\ 0 & 1 & -2 & 5 \end{array}\right] R_2\cdot 3+R_1\to R_1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -4 & 16 \\ 0 & 1 & -2 & 5 \end{array}\right]$$

$$\begin{cases} x-4z=16 \\ y-2z=5 \end{cases}$$

$$\begin{cases} x=16+4z \\ y=5+2z \end{cases}$$

Ans.

$$\boxed{\begin{cases} x=16+4t \\ y=5+2t \\ z=t \end{cases}, \ t\in R}$$

14. (3 marks) Find the point of intersection of the lines L1: $\begin{cases} x=3+t \\ y=6-t \\ z=1+2t \end{cases}$ and L2: $\begin{cases} x=-2+2s \\ y=8-s \\ z=-6+3s \end{cases}$.

$$\begin{cases} 3+t=-2+2s \Rightarrow t=-5+2s \\ 6-t=8-s \\ 1+2t=-6+3s \end{cases}$$

$$6-(-5+2s)=8-s$$

$$11-2s=8-s$$

$$\boxed{3=s}$$

$$\boxed{t=-5+2\cdot 3=1}$$

Check

Eq. 3.

$$1+2\cdot 1=-6+3\cdot 3$$

$$3=3 \text{ ok}$$

L1 and $t=1$

$$\begin{cases} x=3+1=4 \\ y=6-1=5 \\ z=1+2\cdot 1=3 \end{cases}$$

Ans. The point of intersection is $\boxed{(4,5,3)}$

15. (8 marks) Maximize $P = 3x_1 + 2x_2 + 5x_3 + x_4$ subject to $\begin{cases} x_1 + 3x_2 + x_3 + x_4 \le 5 \\ x_1 \qquad\quad - x_3 + 2x_4 \le 2 \\ \quad 2x_2 + x_3 \qquad\quad\;\; \le 3 \end{cases}$.

$(x_1, x_2, x_3, x_4 \ge 0)$

$$\begin{array}{c} \begin{array}{cccccccccc} x_1 & x_2 & x_3 & x_4 & s_1 & s_2 & s_3 & P & & \end{array} \\ \left[\begin{array}{cccc|ccc|c|c} 1 & 3 & 1 & 1 & 1 & 0 & 0 & 0 & 5 \\ 1 & 0 & -1 & 2 & 0 & 1 & 0 & 0 & 2 \\ 0 & 2 & 1 & 0 & 0 & 0 & 1 & 0 & 3 \\ \hline -3 & -2 & \boxed{-5} & -1 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

$5 \div 1 = 5$

$3 \div 1 = 3$ smallest

$R_3 \cdot (-1) + R_1 \to R_1$

$R_3 + R_2 \to R_2$

$R_3 \cdot 5 + R_4 \to R_4$

$$\left[\begin{array}{cccc|ccc|c|c} 1 & 1 & 0 & 1 & 1 & 0 & -1 & 0 & 2 \\ 1 & 2 & 0 & 2 & 0 & 1 & 1 & 0 & 5 \\ 0 & 2 & 1 & 0 & 0 & 0 & 1 & 0 & 3 \\ \hline \boxed{-3} & 8 & 0 & -1 & 0 & 0 & 5 & 1 & 15 \end{array}\right]$$

$2 \div 1 = 2$ smallest

$5 \div 1 = 5$

$R_1 \cdot (-1) + R_2 \to R_2$

$R_1 \cdot 3 + R_4 \to R_4$

$$\begin{array}{c} \begin{array}{cccccccccc} x_1 & x_2 & x_3 & x_4 & & & & P & & \end{array} \\ \left[\begin{array}{cccc|ccc|c|c} 1 & 1 & 0 & 1 & 1 & 0 & -1 & 0 & 2 \\ 0 & 1 & 0 & 1 & -1 & 1 & 2 & 0 & 3 \\ 0 & 2 & 1 & 0 & 0 & 0 & 1 & 0 & 3 \\ \hline 0 & 11 & 0 & 2 & 3 & 0 & 2 & 1 & 21 \end{array}\right] \end{array}$$

$\underbrace{0 \quad 11 \quad 0 \quad 2}_{\text{no negative numbers}}$

$$\boxed{\begin{array}{l} x_1 = 2 \\ x_2 = 0 \\ x_3 = 3 \\ x_4 = 0 \\ P = 21 \end{array}}$$