DAWSON COLLEGE
MATHEMATICS DEPARTMENT
Applied Mathematics

201-923-DW
Fall 2024
Final Examination
December 17th, 2024
Time Limit: 3 hours

Name: Worked solutions

ID#:

Instructor: M. Hitier

- ▷ This exam contains 11 pages (including this cover page) and 12 problems. Check to see if any pages are missing.
- ▷ Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page, and please indicate that you have done so.
- ▷ Give the work in full; – unless otherwise stated, reduce each answer to its simplest, exact form; – and write and arrange your exercise in a legible and orderly manner.
- ▷ You are only permitted to use the **Sharp EL-531**** calculator.
- ▷ This examination booklet must be returned intact.
- ▷ Good luck!

Question	Points	Score
1	15	
2	10	
3	6	
4	8	
5	14	
6	8	
7	12	
8	7	
9	4	
10	6	
11	6	
12	4	
Total:	100	

1. If $A = \begin{bmatrix} -1 & 3 & -2 \\ 2 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -2 \\ 1 & 0 \\ 3 & -1 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix}$, find the following

[5 pts] (a) $3A - 2B^T$

$$= \begin{bmatrix} -3 & 9 & -6 \\ 6 & 0 & 3 \end{bmatrix} - 2\begin{bmatrix} 3 & 1 & 3 \\ -2 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 9 & -6 \\ 6 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 6 & 2 & 6 \\ -4 & 0 & -2 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} -9 & 7 & -12 \\ 10 & 0 & 5 \end{bmatrix}}$$

[5 pts] (b) $C + BA$

2×2

$$BA = \begin{bmatrix} 3 & -2 \\ 1 & 0 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -2 \\ 2 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3-4 & 9+0 & -6-2 \\ -1+0 & 3+0 & -2+0 \\ -3-2 & 9+0 & -6-1 \end{bmatrix}$$

3×3

Impossible matrices of different size

[5 pts] (c) C^2

$$C^2 = \begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix} = \begin{bmatrix} 1-6 & 2+0 \\ -3+0 & -6+0 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} -5 & 2 \\ -3 & -6 \end{bmatrix}}$$

[7 pts] 2. (a) Find M^{-1} for $M = \begin{bmatrix} 1 & -2 & 2 \\ 2 & -3 & 4 \\ 1 & -1 & 4 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 2 & -3 & 4 & 0 & 1 & 0 \\ 1 & -1 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \end{array}\right] \xrightarrow[R_3 - R_2]{} \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 2 & 1 & -1 & 1 \end{array}\right]$$

$$\xrightarrow[\frac{1}{2}R_3]{} \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1/2 & -1/2 & 1/2 \end{array}\right] \xrightarrow{R_1 - 2R_3} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1/2 & -1/2 & 1/2 \end{array}\right]$$

$$\xrightarrow[R_1 + 2R_2]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & 3 & -1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1/2 & -1/2 & 1/2 \end{array}\right]$$

$$\boxed{M^{-1} = \begin{bmatrix} -4 & 3 & -1 \\ -2 & 1 & 0 \\ 1/2 & -1/2 & 1/2 \end{bmatrix}}$$

[3 pts] (b) Use your previous answer to solve $\begin{cases} x - 2y + 2z = -3 \\ 2x - 3y + 4z = 0 \\ x - y + 4z = 2 \end{cases} \iff M \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}$

$$\iff \begin{bmatrix} x \\ y \\ z \end{bmatrix} = M^{-1} \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 12 + 0 - 2 \\ 6 + 0 + 0 \\ -3/2 + 0 + 1 \end{bmatrix}$$

$$= \begin{bmatrix} 10 \\ 6 \\ -1/2 \end{bmatrix}$$

$$\boxed{\begin{aligned} x &= 10 \\ y &= 6 \\ z &= -1/2 \end{aligned}}$$

[6 pts] 3. Use systems of equations (Gauss-Jordan or Gaussian elimination) to balance the following chemical reaction

No marks if another method is used (e.g., inspection, not using matrices).

$$x\,Fe_2SiO_4 + y\,O_2 \longrightarrow z\,Fe_2O_3 + w\,FeSiO_3$$

$$\begin{aligned}
Fe&: 2x = 2z + w \Rightarrow 2x + 0y - 2z - w = 0\\
Si&: x = w \Rightarrow x + 0y + 0z - w = 0\\
O&: 4x + 2y = 3z + 3w \Rightarrow 4x + 2y - 3z - 3w = 0
\end{aligned}$$

$$\left[\begin{array}{cccc|c} 2 & 0 & -2 & -1 & 0\\ 1 & 0 & 0 & -1 & 0\\ 4 & 2 & -3 & -3 & 0 \end{array}\right] \xrightarrow[R_3-2R_1]{} \left[\begin{array}{cccc|c} 2 & 0 & -2 & -1 & 0\\ 1 & 0 & 0 & -1 & 0\\ 0 & 2 & 1 & -1 & 0 \end{array}\right]$$

$$\xrightarrow[R_1\leftrightarrow R_2]{} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0\\ 2 & 0 & -2 & -1 & 0\\ 0 & 2 & 1 & -1 & 0 \end{array}\right] \xrightarrow[R_2-2R_1]{} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0\\ 0 & 0 & -2 & 1 & 0\\ 0 & 2 & 1 & -1 & 0 \end{array}\right]$$

$$\xrightarrow[R_2\leftrightarrow R_3]{} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0\\ 0 & 2 & 1 & -1 & 0\\ 0 & 0 & -2 & 1 & 0 \end{array}\right] \xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0\\ 0 & 1 & 1/2 & -1/2 & 0\\ 0 & 0 & -2 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{2}R_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0\\ 0 & 1 & 1/2 & -1/2 & 0\\ 0 & 0 & 1 & -1/2 & 0 \end{array}\right] \xrightarrow{R_2-\frac{1}{2}R_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0\\ 0 & 1 & 0 & -1/4 & 0\\ 0 & 0 & 1 & -1/2 & 0 \end{array}\right]$$

$$\left\{\begin{array}{l} w = t\\ z - \frac{1}{2}t = 0 \Rightarrow z = \frac{1}{2}t\\ y - \frac{1}{4}t = 0 \Rightarrow y = \frac{1}{4}t\\ x - t = 0 \Rightarrow x = t \end{array}\right. \xrightarrow[t=4]{\text{take}} \left\{\begin{array}{l} w = 4\\ z = 2\\ y = 1\\ x = 4 \end{array}\right.$$

$$\boxed{4\,FeSiO_4 + O_2 \longrightarrow 2\,Fe_2O_3 + 4\,FeSiO_3}$$

[8 pts] 4. Find the determinant of $A = \begin{bmatrix} 1 & -2 & 1 & 3 \\ 2 & 0 & -1 & 3 \\ 0 & 5 & 2 & 0 \\ 1 & -2 & 0 & 4 \end{bmatrix}$

Along R_3: $\det(A) = -5 \begin{vmatrix} 1 & 1 & 3 \\ 2 & -1 & 3 \\ 1 & 0 & 4 \end{vmatrix} + 2 \begin{vmatrix} 1 & -2 & 3 \\ 2 & 0 & 3 \\ 1 & -2 & 4 \end{vmatrix}$

$-3 \quad 0 \quad 8 \to 5$

$\begin{vmatrix} 1 & 1 & 3 \\ 2 & -1 & 3 \\ 1 & 0 & 4 \end{vmatrix} \begin{matrix} 1 & 1 \\ 2 & -1 \\ 1 & 0 \end{matrix}$

$-4 \quad 3 \quad 0 \to -1$

$= -5(-1-5) + 2(-18+22)$

$= (-5)(-6) + 2(4)$

$= 30 + 8 = \boxed{38}$

$0 \quad -6 \quad -16 \to -22$

$\begin{vmatrix} 1 & -2 & 3 \\ 2 & 0 & 3 \\ 1 & -2 & 4 \end{vmatrix} \begin{matrix} 1 & -2 \\ 2 & 0 \\ 1 & -2 \end{matrix}$

$0 \quad -6 \quad -12 \to -18$

5. Given the force $F_1 = 4$ Newtons 30^o East of North, $F_2 = 5$ Newtons West and $F_3 = 2$ Newtons 45^o East of South.

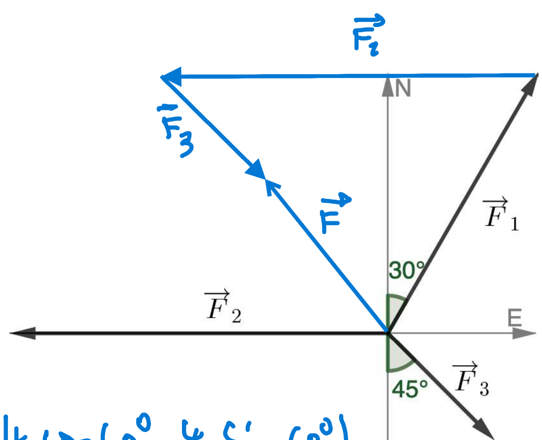
[3 pts] (a) Using only the diagram below (no computation), sketch the resultant force $\overrightarrow{F} = \overrightarrow{F_1} + \overrightarrow{F_2} + \overrightarrow{F_3}$ below. Show your construction clearly.

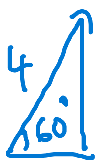
(b) Compute numerically (i.e. do not read the value on the diagram, but use the informations in the text):

[6 pts] i. The components of each of the forces $\overrightarrow{F_1}$, $\overrightarrow{F_2}$, and $\overrightarrow{F_3}$. *–Give exact answers–*

[2 pts] ii. The components of the resultant force $\overrightarrow{F}$. *–Round to 3 decimal places–*

[3 pts] iii. The magnitude and direction of the resultant force $\overrightarrow{F}$. *–Round to 3 decimal places–*



(i) 

$\vec F_1 = (4\cos 60^\circ, 4\sin 60^\circ)$

$= (4\cdot\frac{1}{2}, 4\frac{\sqrt{3}}{2}) = \boxed{(2, 2\sqrt{3})}$

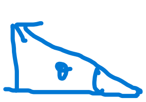
$\boxed{\vec F_2 = (-5, 0)}$

$\vec F_3 = (2\frac{\sqrt{2}}{2}, -2\frac{\sqrt{2}}{2}) = \boxed{(\sqrt{2}, -\sqrt{2})}$

(ii) $\vec F = \vec F_1 + \vec F_2 + \vec F_3 = (2-5+\sqrt{2}, 2\sqrt{3}+0-\sqrt{2}) = (-3+\sqrt{2}, 2\sqrt{3}-\sqrt{2})$

$\approx \boxed{(-1.586, 2{,}050)}$

(iii) $\|\vec F\| = \sqrt{(-3+\sqrt{2})^2 + (2\sqrt{3}-\sqrt{2})^2} \approx \boxed{2.592}$

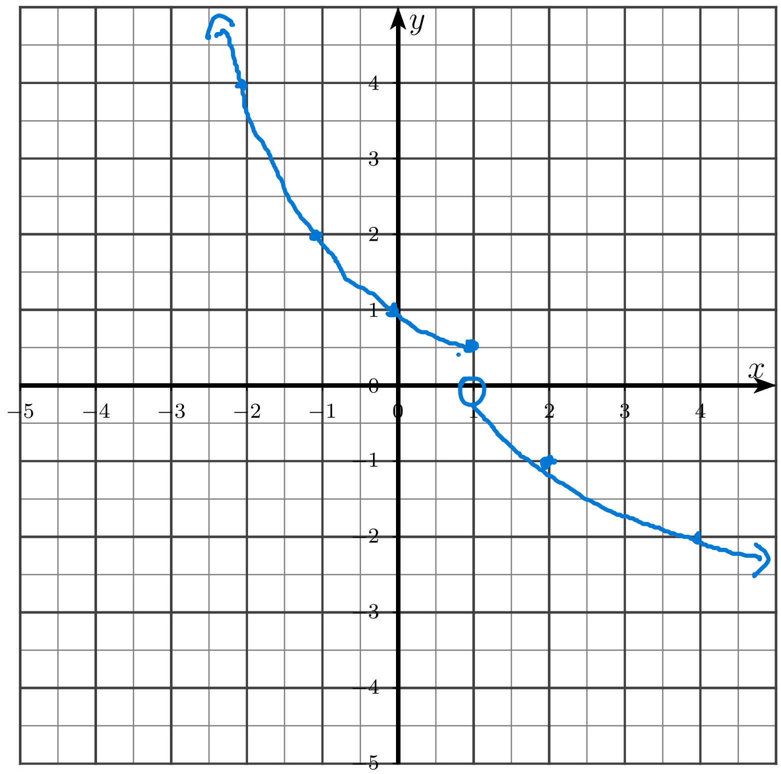


$\tan\theta = \frac{2\sqrt{3}-\sqrt{2}}{-3+\sqrt{2}} \Rightarrow \theta = \mathrm{Arctan}\left(\frac{2\sqrt{3}+\sqrt{2}}{-3+\sqrt{2}}\right) \times \frac{180}{\pi} \approx 37{,}725^0$

$\Rightarrow$ Direction: $\boxed{37{,}725^0 \text{ north of west}}$

[4 pts] 6. (a) Using some well chosen points, graph the function

$$f(x) = \begin{cases} \left(\frac{1}{2}\right)^x & \text{if } x \leq 1 \\ -\log_2 x & \text{if } x > 1 \end{cases}$$



[2 pts] (b) State its domain

$\mathbb{R}$

[2 pts] (c) State its range

$(-\infty, 0) \cup [1, \infty)$

7. Solve for x

[6 pts] (a) $\left(\frac{1}{2}\right)^{2x+3} = 8^{x-5}$

$(2^{-1})^{(2x+3)} = (2^3)^{x-5}$

$2^{-2x-3} = 2^{3x-15}$

$-2x-3 = 3x-15$

$+2x \quad +15 \qquad +2x \quad +15$

$12 = 5x$

$\boxed{x = \frac{12}{5}}$

[6 pts] (b) $\ln(x) + \ln(2x+1) = 0$

$\ln(x(2x+1)) = \ln 1$

$2x^2 + x = 1$

$2x^2 + x - 1 = 0$

$(2x-1)(x+1) = 0$

$2x - 1 = 0 \qquad x+1=0$

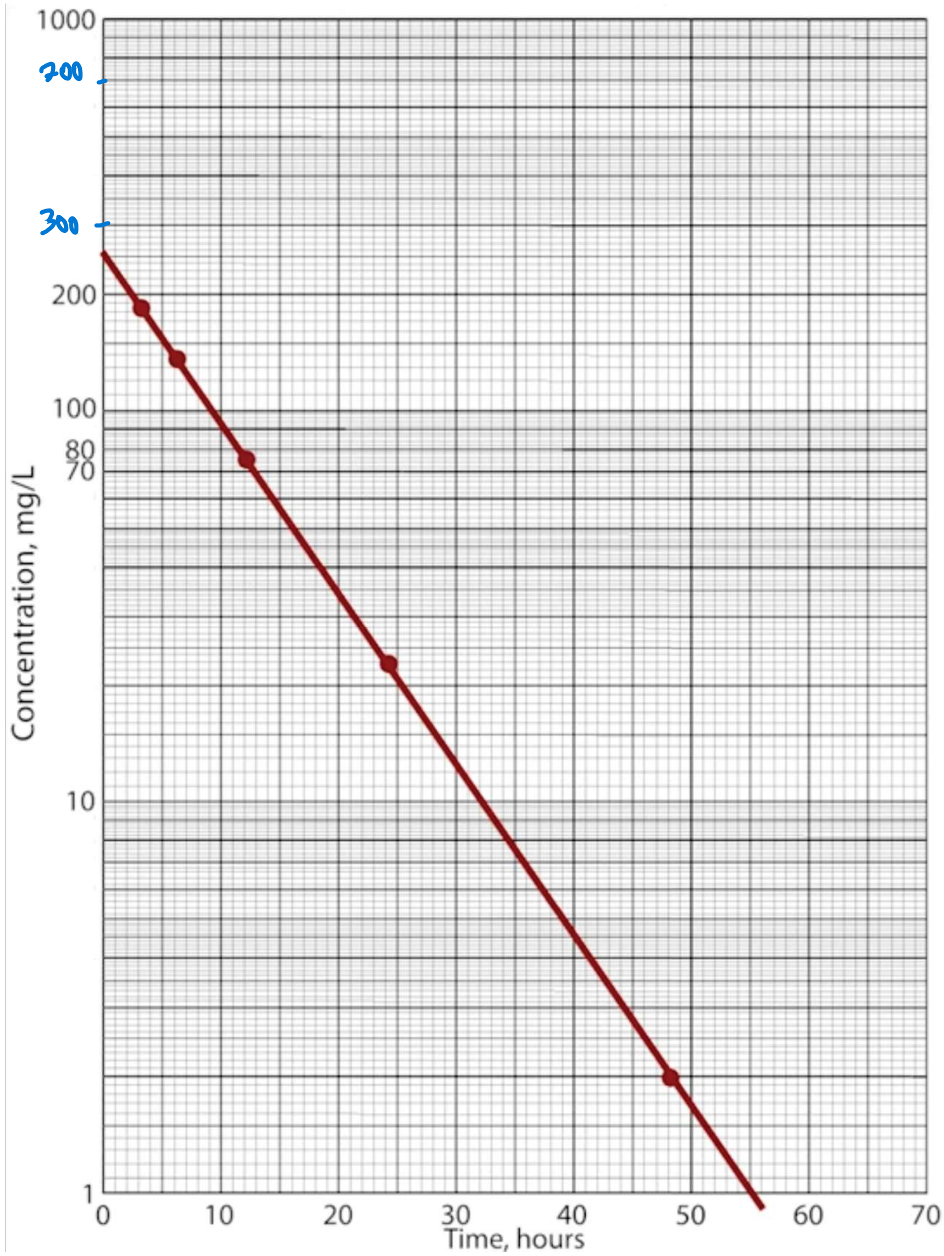
$\boxed{x = \frac{1}{2}} \qquad \cancel{x = -1}$

check: $x = \frac{1}{2}$: $\ln(\frac{1}{2}) + \ln(2\cdot\frac{1}{2}+1) = \ln(1/2) + \ln 2 = 0$

$x = -1$ $\ln(x)$ DNE $\Rightarrow$ extraneous

8. The following set of data has been plotted on a semi-logarithmic paper:

Time (x **hours**)	**Concentration** (y **mg/L**)
3	185.2
6	137.2
12	75.30
24	22.68
48	2.057



[2 pts] (a) Which functional pattern does the data follow?

☐ a linear pattern $y = ax + b$ ☑ an exponential pattern $y = b \cdot a^x$ ☐ a power pattern $y = b \cdot x^a$

[2 pts] (b) Read an approximate initial concentration on the graph.

250 mg/l

[3 pts] (c) Given that the slope of the line is approximately -0.04343, determine an approximate value of the parameters a and b.

$b = 250$ $a = 10^{-0.04343} \approx 0.9048$

$$\boxed{y = 250 \cdot 0.9048^x}$$

[4 pts] 9. Write in terms of the simplest logarithm possible

$\log\left(\frac{10\sqrt{x}}{x^2-1}\right) = \log 10 + \log x^{1/2} - \log(x^2-1)$

$= 1 + \frac{1}{2}\log x - (\log(x-1) + \log(x+1))$

$= \boxed{1 + \frac{1}{2}\log x - \log(x-1) - \log(x+1)}$

[6 pts] 10. Verify the identity

$\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

LHS: $\tan(A+B) = \dfrac{\sin(A+B)}{\cos(A+B)}$

$= \dfrac{\sin A\cos B + \sin B\cos A}{\cos A\cos B - \sin A\sin B}$

RHS $= \dfrac{\frac{\sin A}{\cos A} + \frac{\sin B}{\cos B}}{1 - \frac{\sin A}{\cos A}\frac{\sin B}{\cos B}}$

$= \dfrac{\frac{\sin A\cos B + \sin B\cos A}{\cos A\cos B}}{\frac{\cos A\cos B - \sin A\sin B}{\cos A\cos B}}$ $\times \cos A\cos B$ $\times \cos A\cos B$

$= \dfrac{\sin A\cos B + \sin B\cos A}{\cos A\cos B - \sin A\sin B}$

$=$

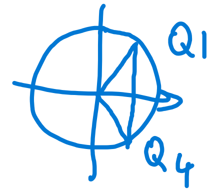
LHS = RHS ✓

[6 pts] 11. Solve the equation $\sin(2x) = \sin(x)$ for $0 \le x < 2\pi$

$2\sin x \cos x = \sin x$

$2\sin x \cos x - \sin x = 0$

$\sin x(2\cos x - 1) = 0$



$\sin x = 0$

$\boxed{x = 0 \text{ or } x = \pi}$

$2\cos x - 1 = 0$

$\cos x = \frac{1}{2}$

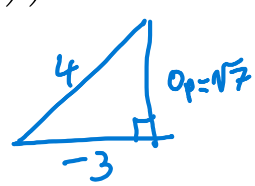
$x = \boxed{\frac{\pi}{3}}$

or $x = 2\pi - \frac{\pi}{3} = \boxed{\frac{5\pi}{3}}$

QII $\sin\theta > 0$

[4 pts] 12. Find the exact value of $\sin\left(\arccos\left(-\frac{3}{4}\right)\right)$

$\cos\theta = -\frac{3}{4}$



$4^2 = Op^2 + (-3)^2$

$16 = Op^2 + 9$

$7 = Op^2$

$\sin\left(\arccos\left(-\frac{3}{4}\right)\right) = \sin\theta = \boxed{\frac{\sqrt{7}}{4}}$