DAWSON COLLEGE

MATHEMATICS DEPARTMENT

Engineering Math II

201-942-DW 1
Fall 2017
Final Exam
December 18, 2017
Time Limit: 3 hours

Name: Solutions.

ID#: ____________________

- This exam contains 12 pages (including this cover page) and 16 problems. Check to see if any pages are missing.
- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page, and please indicate that you have done so.
- Give the work in full; – unless otherwise stated, reduce each answer to its simplest, exact form; – and write and arrange your exercise in a legible and orderly manner.
- You are only permitted to use the **Sharp EL-531XG** or **Sharp EL-531X** calculator.
- This examination booklet must be returned intact.
- Good luck!

Question	Points	Score
1	8	
2	5	
3	12	
4	12	
5	5	
6	5	
7	5	
8	5	
9	8	
10	10	
11	5	
12	5	
13	5	
14	5	
15	5	
Total:	100	

1. Find each of the following limits.

   (a) (4 points) $\lim_{x \to 4} \frac{x^2 - 16}{x - 4}$ $= \lim_{x \to 4} \frac{(x+4)\cancel{(x-4)}}{\cancel{x-4}} = \lim_{x \to 4} (x+4) = 4+4 = 8$

   (b) (4 points) $\lim_{x \to 4} \frac{\frac{1}{4} - \frac{1}{x}}{x^2 - x - 12}$ $= \lim_{x \to 4} \frac{\frac{\cancel{x-4}}{4x}}{\cancel{(x-4)}(x+3)} = \lim_{x \to 4} \frac{1}{4x(x+3)}$

   $= \frac{1}{4(4)(4+3)} = \frac{1}{112}$

2. (5 points) Using only the limit definition of the derivative, find the derivative of $f(x) = x^2 + 4x - 1$.

$f'(x) = \lim_{h \to 0} \frac{\left((x+h)^2 + 4(x+h) - 1\right) - \left(x^2 + 4x - 1\right)}{h}$

$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 + \cancel{4x} + 4h \cancel{-1} - \cancel{x^2} - \cancel{4x} + \cancel{1}}{h}$

$= \lim_{h \to 0} \frac{\cancel{h}(2x+h+4)}{\cancel{h}} = \lim_{h \to 0} (2x+h+4) = 2x + 4$

3. Find y' for each of the following. Do not simplify.

(a) (4 points) $y = (x+1)^2(x^2+2x)^3$

$$y' = 2(x+1)(x^2+2x)^3 + (x+1)^2(3)(x^2+2x)^2(2x+2)$$

(b) (4 points) $y = (2x+1)\sqrt{2x+1}$

$$y' = 2\cdot(2x+1)^{1/2} + (2x+1)\left(\tfrac{1}{2}\right)(2x+1)^{-1/2}\cdot(2)$$

(c) (4 points) $y = \left(\dfrac{x}{x-1}\right)^2$

$$y' = 2\left(\frac{x}{x-1}\right)\cdot\left(\frac{(x-1)(1) - x(1)}{(x-1)^2}\right)$$

4. Consider the function given by $f(x) = 2x^3 - 3x^2 - 12x + 20$.

(a) (4 points) Find the equations of the tangent line and normal line to $f(x)$ at the point where $x = 1$.

$f'(x) = 6x^2 - 6x - 12$

$y = f(1) = 2 - 3 - 12 + 20 = 7$

$m_1 = f'(1) = 6 - 6 - 12 = -12.$

Tangent

$y - 7 = -12(x-1)$

$y = -12x + 12 + 7$

$\boxed{y = -12x + 19}$

$m_2 = \frac{-1}{m_1} = \frac{1}{12}$, Normal

$y - 7 = \frac{1}{12}(x-1)$

$y = \frac{1}{12}x - \frac{1}{12} + 7$

$\boxed{y = \frac{1}{12}x + \frac{83}{12}}$

(b) (4 points) Find all points on $f(x)$ that have a horizontal tangent line.

$f'(x) = 0 \Rightarrow 0 = 6(x^2 - x - 2) = 6(x-2)(x+1)$

$\therefore$ $x = 2$ and $x = -1$

(c) (4 points) Find all maximums and minimums of $f(x)$.

x		-1		2	
y'	$+$		$-$		$+$

$y'(-2) = 6(-4)(-1) > 0$

$y'(0) = -12 < 0$

$y'(3) = 6(1)(4) > 0$

$f(-1) = -2 - 3 + 12 + 20 = 27$ $\therefore$ max at $(-1, 27)$

$f(2) = 2(8) - 3(4) - 24 + 20 = 0$ $\therefore$ min at $(2, 0)$.

5. (5 points) Find $\frac{dy}{dx}$ given the implicit relation

$$x^3 + 4xy - 3y^{4/3} = 2x$$

$$3x^2 + 4((1)y + xy') - 3\left(\frac{4}{3}\right)y^{1/3} \cdot y' = 2$$

$$3x^2 + 4y + 4xy' - 4y^{1/3}y' = 2$$

$$y'(4x - 4y^{1/3}) = 2 - 3x^2 - 4y$$

$$y' = \frac{2 - 3x^2 - 4y}{4x - 4y^{1/3}}$$

6. (5 points) A 45-caliber bullet fired straight up from the surface of the moon would reach a height of $s = 832t - 2.6t^2$ feet after t sec. On Earth, in the absence of air, its height would be $s = 832t - 16t^2$ feet after t sec. How long will the bullet be aloft in each case? How high will the bullet go in each case?

Moon $S = 832t - 2.6t^2$

$V = 832 - 5.2t$

$V = 0 \Rightarrow t = \frac{832}{5.2} = 160s$

$S = 0 \Rightarrow 2.6t(320 - t) = 0$

$\therefore t = 0$ or $\boxed{t = 320s}$

$S(160) = 832(160) - 2.6(160)^2$

$= 66560 ft.$

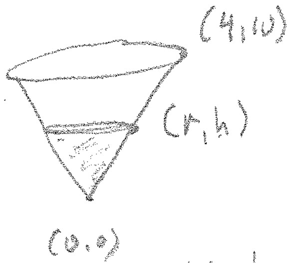
Earth $S = 0 \Rightarrow 16t(52 - t) = 0$

$\therefore t = 0$ or $t = 52s$

$S(26) = 832(26) - 16(26)^2$

$= 10816 ft$

7. (5 points) Water drains from a conical tank of radius 4 feet and height 10 feet at the rate of 5 ft^3/min. How fast is the water level dropping when $h = 6$ feet? $\left(V = \frac{1}{3}\pi r^2 h\right)$



$\frac{r}{4} = \frac{h}{10}$ $\qquad \frac{dV}{dt} = -5\ ft^3/min$

$\frac{5}{2}r = h$ $\qquad \frac{dh}{dt} = ?$ when $h = 6$ ft

$r = \frac{2}{5}h$

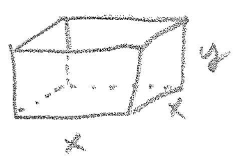
$V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi\left(\frac{2}{5}h\right)^2 h = \frac{1}{3}\pi\left(\frac{4}{25}\right)h^3 = \frac{4}{75}\pi h^3$

$V' = \frac{4}{75}\pi(3h^2)\cdot h' = \frac{4}{25}\pi h^2 \cdot h'$

$\therefore h' = \frac{25V'}{4\pi h^2} = \frac{25(-5)}{4\pi(6)^2} = \frac{-125}{144\pi}$ ft/min ≈ -0.276 ft/min.

8. (5 points) A customer has asked you to design an open-top rectangular stainless steel vat. It is to have a square base and a volume of 32 ft^3, and to weigh no more than necessary. What dimensions do you recommend?



$V = \boxed{x^2 y = 32}$ constraint $\Rightarrow \boxed{y = \frac{32}{x^2}}$

minimize Area = base + 4 sides

$= x^2 + 4(xy)$

$\therefore A = x^2 + 4xy$

$A = x^2 + 4x\left(\frac{32}{x^2}\right) = x^2 + 128x^{-1}$

$A' = 2x - 128x^{-2} = 0$

$2x = \frac{128}{x^2}$

$x^3 = 64$

$\underline{x = 4}$

9. Find each indefinite integral.

(a) (4 points) $\int \left(\frac{x^3 + 5x - \sqrt{x}}{x^2} \right)\, dx$

$$= \int \left(\frac{x^3}{x^2} + \frac{5x}{x^2} - \frac{\sqrt{x}}{x^2} \right) dx = \int \left(x + \frac{5}{x} - x^{-3/2} \right) dx$$

$$= \frac{x^2}{2} + 5 \ln|x| - \frac{x^{-1/2}}{-1/2} + C$$

$$= \frac{1}{2}x^2 + 5 \ln|x| + \frac{2}{\sqrt{x}} + C$$

(b) (4 points) $\int \frac{x}{\sqrt{7 + x^2}}\, dx$

$u = 7 + x^2$

$du = 2x\,dx$

$$= \frac{1}{2} \int \frac{2x\,dx}{\sqrt{7+x^2}} = \frac{1}{2} \int \frac{du}{\sqrt{u}} = \frac{1}{2} \int u^{-1/2}\, du$$

$$= \cancel{\tfrac{1}{2}} \left(\frac{u^{1/2}}{\cancel{1/2}} \right) + C = \sqrt{7+x^2} + C$$

10. Find each definite integral.

(a) (5 points) $\int_0^1 (8x^3 - 12x^2 + 5)\ dx$

$$= \left(\frac{8x^4}{4} - \frac{12x^3}{3} + 5x \right) \Big|_0^1 = \left(2x^4 - 4x^3 + 5x \right) \Big|_0^1$$

$$= (2 - 4 + 5) - (0)$$

$$= 3$$

(b) (5 points) $\int_0^{1/2} x^3(1 + 9x^4)^{-3/2}\ dx$

$u = 1 + 9x^4$ $\quad x = 0 \Rightarrow u = 1$

$du = 36x^3 dx$ $\quad x = \frac{1}{2} \Rightarrow u = 1 + 9\left(\frac{1}{2}\right)^4 = 25/16$

$$= \frac{1}{36} \int_0^{1/2} 36x^3 (1 + 9x^4)^{-3/2}\ dx$$

$$= \frac{1}{36} \int_1^{25/16} u^{-3/2}\ du = \frac{1}{36} \left(\frac{u^{-1/2}}{-1/2} \right) \Bigg|_1^{25/16} = \frac{1}{36} \left(\frac{-2}{1} \right) \left(\frac{1}{\sqrt{u}} \right) \Bigg|_1^{25/16}$$

$$= -\frac{1}{18} \left(\frac{1}{\sqrt{25/16}} - \frac{1}{\sqrt{1}} \right) = -\frac{1}{18} \left(\frac{4}{5} - 1 \right) = \frac{1}{90} \approx 0.01111\ldots$$

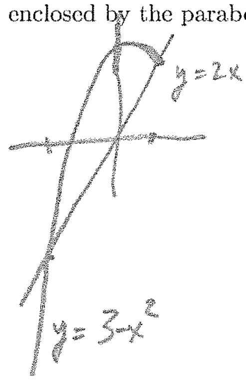
11. (5 points) Find the area of the region enclosed by the parabola $y = 3 - x^2$ and the line $y = 2x$.

$3 - x^2 = 2x$

$0 = x^2 + 2x - 3$

$0 = (x+3)(x-1)$

$x = -3$ or $x = 1$



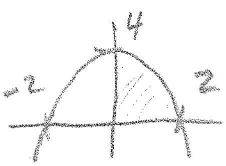
$$A = \int_{-3}^{1} (3 - x^2 - 2x)\,dx$$

$$= \left(3x - \frac{x^3}{3} - x^2\right)\Big|_{-3}^{1}$$

$$= \left(3 - \frac{1}{3} - 1\right) - \left(-9 + \frac{27}{3} - 9\right)$$

$$= \frac{5}{3} + 9 = \frac{32}{3}$$

12. (5 points) Find $\bar{x}$ and $\bar{y}$, the coordinates of the centroid of the region bounded by the parabola $y = 4 - x^2$ and the x-axis.



By symmetry, $\bar{x} = 0$, and look only at 1st quadrant.

$$A = \int_0^2 (4 - x^2)\,dx = \left(4x - \frac{x^3}{3}\right)\Big|_0^2 = 8 - \frac{8}{3} = \frac{16}{3}$$

$$\bar{y} = \frac{1}{2A}\int_a^b (y_1 + y_2)(y_2 - y_1)\,dx = \frac{1}{2\left(\frac{16}{3}\right)}\int_0^2 (4 - x^2)(4 - x^2)\,dx$$

$$= \frac{3}{32}\int_0^2 (16 - 8x^2 + x^4)\,dx = \frac{3}{32}\left(16x - \frac{8x^3}{3} + \frac{x^5}{5}\right)\Big|_0^2$$

$$= \frac{3}{32}\left(32 - \frac{8(8)}{3} + \frac{32}{5}\right) = \frac{3}{32}\left(\frac{\cancel{64}\ 256}{15}\right) = \frac{\cancel{2}\ 8}{5} = 8/5$$

$\bar{y} = \cancel{0.4} = 1.6$

13. (5 points) Determine the arc length of $y = \frac{1}{3}(x^2+2)^{3/2}$ from $x = 0$ to $x = 3$.

$$y' = \frac{1}{\cancel{3}} \cdot \frac{\cancel{3}}{\cancel{2}}(x^2+2)^{1/2} \cdot (\cancel{2}x) = x \cdot \sqrt{x^2+2}$$

$$1+(y')^2 = 1+\left(x\sqrt{x^2+2}\right)^2 = 1 + x^2(x^2+2) = x^4+2x^2+1$$

$$= (x+1)^2.$$

$$S = \int_0^3 \sqrt{1+(y')^2}\,dx = \int_0^3 \sqrt{(x+1)^2}\,dx = \int_0^3 (x+1)\,dx$$

$$= \left(\frac{x^2}{2}+2x\right)\Big|_0^3 = \left(\frac{9}{2}+6\right) - (0) = \frac{21}{2}$$

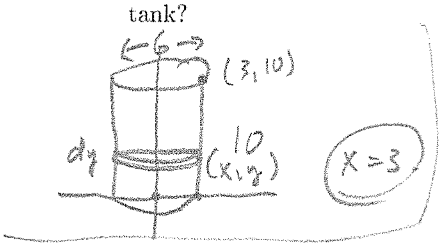
14. (5 points) If a force of 90 N stretches a spring 1 m beyond its natural length, how much work does it take to stretch the spring 5 m beyond its natural length?

$F = kx \Rightarrow 90 = k \cdot 1 \Rightarrow k = 90 \text{ N/m} \cdot$ and $F = 90x$

$W = \int_0^5 (90x)\,dx = \frac{90x^2}{2}\Big|_0^5 = 45x^2\Big|_0^5$

$= 45(25-0) = 1125 \text{ N} \cdot \text{m}.$

15. (5 points) A vertical right circular cylindrical tank measures 10 m high and 6 m in diameter. It is full of kerosene with mass density 820.1 kg/m^3. How much work does it take to pump the kerosene to the level of the top of the tank?



$W = \int_0^{10} (10-y) \cdot (g) \cdot (820.1)\pi x^2 \cdot dy$

$= 820.1 g\pi \int_0^{10} (10-y) \cdot (3)^2\,dy$

$= 7390.9 g\pi \int_0^{10} (10-y)\,dy = 7380.9 g\pi \left(10y - y^2/2\right)\Big|_0^{10}$

$= 7380.9 g\pi \left(10(10) - 10^2/2\right) = 7380.9\, g\pi\,(50)$

$\approx 11{,}400{,}000 \text{ N} \cdot \text{m}.$

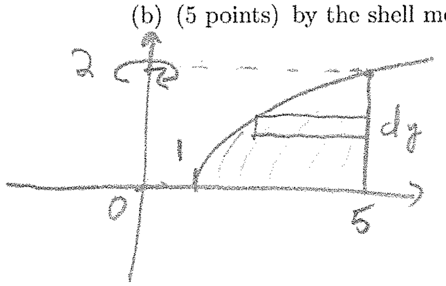
BONUS QUESTION (Do this only if you have completed the rest of the test.)

16. Consider the region bounded by $x = y^2 + 1$, $x = 5$, and the x-axis. Find the volume of the solid generated by revolving this region around the y-axis

    (a) (5 points) by the washer method,

    (b) (5 points) by the shell method.

a)



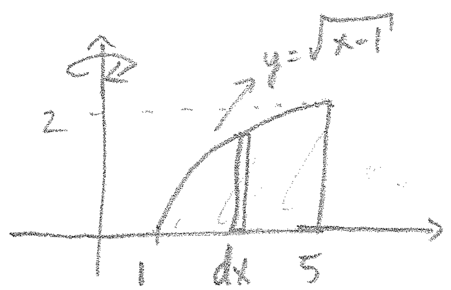
$$V = \pi \int_0^2 \left(5^2 - (y^2+1)^2\right) dy$$

$$= \pi \int_0^2 \left(25 - (y^4 + 2y^2 + 1)\right) dy$$

$$= \pi \int_0^2 (-y^4 - 2y^2 + 24)\,dy = \pi \left(-\frac{y^5}{5} - \frac{2y^3}{3} + 24y\right)\Bigg|_0^2$$

$$= \pi\left[\left(\frac{-2^5}{5} - \frac{2(2^3)}{3} + 24(2)\right) - \left(0\right)\right] = \pi\left(\frac{544}{15}\right) = \frac{544\pi}{15} \approx 113.9$$

b)



$$V = 2\pi \int_1^5 x\left(\sqrt{x-1}\right) dx$$

$u = x-1 \quad du = dx$

$x = u+1$

$x = 1 \rightarrow u = 0$

$x = 5 \rightarrow u = 4$

$$= 2\pi \int_0^4 (u+1)\cdot\sqrt{u}\,du$$

$$= 2\pi \int_0^4 \left(u^{3/2} + u^{1/2}\right) du = 2\pi\left(\frac{u^{5/2}}{5/2} + \frac{u^{3/2}}{3/2}\right)\Bigg|_0^4$$

$$= 4\pi\left[\left(\frac{1}{5}(4)^{5/2} + \frac{1}{3}(4)^{3/2}\right) - \left(0\right)\right]$$

$$= 4\pi\left(\frac{136}{15}\right) = \frac{544\pi}{15} \approx 113.9$$