Dawson College

Mathematics Department

FINAL EXAMINATION FOR FALL 2019

APPLIED MATHEMATICS

201-943-DW

December 19, 2019 2:00pm - 5:00pm

Student Name: Solutions

Student I.D. #: ____________________

Instructor: Alexander Hariton

Time: 3 hours

Instructions:

- Print your name and student ID number in the space provided above.
- All questions are to be answered directly on the examination paper in the space provided. Show your complete work and give explanations.
- A Sharp EL-531XG, Sharp EL-531XT or Sharp EL-531X calculator is permitted.

This examination consists of 20 questions. Please ensure that you have a complete examination.

This examination must be returned intact.

1. (12 marks)

Solve the following equations for x

(a) $8 + 2(x+5) = -5 - (x-2)$

$$8 + 2x + 10 = -5 - x + 2$$

$$2x + 18 = -x - 3$$

$$3x = -21$$

$$\boxed{x = -7}$$

(b) $x^2 - 6x + 7 = 0$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(7)}}{2(1)}$$

$$= \frac{6 \pm \sqrt{36-28}}{2} = \frac{6 \pm \sqrt{8}}{2} = \frac{6 \pm 2\sqrt{2}}{2}$$

$$= \frac{2(3 \pm \sqrt{2})}{2} = 3 \pm \sqrt{2}$$

Solutions : $\boxed{x = 3 + \sqrt{2}}$, $\boxed{x = 3 - \sqrt{2}}$

(c) $x - \sqrt{2x-6} = 3$

$$x - 3 = \sqrt{2x-6}$$

$$(x-3)^2 = (\sqrt{2x-6})^2$$

$$x^2 - 6x + 9 = 2x - 6$$

$$x^2 - 8x + 15 = 0$$

$$(x-3)(x-5) = 0, \quad \text{so} \quad \underline{x=3} \quad \text{or} \quad \underline{x=5}$$

Check: $\underline{x=3}$: $3 - \sqrt{2(3)-6} = 3$, so $3 - \sqrt{0} = 3$, so $3 = 3$ ✓

$\underline{x=5}$: $5 - \sqrt{2(5)-6} = 3$, so $5 - \sqrt{4} = 3$, so $5 - 2 = 3$, $3 = 3$ ✓

Solutions: $\boxed{x=3}$, $\boxed{x=5}$

2. (5 marks)

Perform the following operation on rational expressions. Simplify your answer.

$$\frac{x^3+4x^2}{2x^2+5x-3} \div \frac{x^2-16}{x^2-x-12}$$

$$= \frac{x^3+4x^2}{2x^2+5x-3} \cdot \frac{x^2-x-12}{x^2-16} = \frac{x^2\cancel{(x+4)}}{(2x-1)\cancel{(x+3)}} \cdot \frac{\cancel{(x+3)}\cancel{(x-4)}}{\cancel{(x+4)}\cancel{(x-4)}}$$

$$= \boxed{\frac{x^2}{2x-1}}$$

3. (4 marks)

Solve the following inequality for x. Write your answer in interval notation.

$-1 < 2x + 5 \leq 9$

$-1 - 5 < 2x \leq 9 - 5$

$-6 < 2x \leq 4$

$\underline{-3 < x \leq 2}$

Solution set: $\boxed{(-3, 2]}$

4. (4 marks)

Perform the following polynomial division

$$\frac{x^3 - x^2 - 5x + 9}{x - 3}$$

$$\begin{array}{r} x^2 + 2x + 1 \\ x-3 \,\big)\, \overline{x^3 - x^2 - 5x + 9} \\ \underline{-(x^3 - 3x^2)} \quad\quad\quad \\ 2x^2 - 5x \quad\; \\ \underline{-(2x^2 - 6x)} \quad\; \\ x + 9 \\ \underline{-(x - 3)} \\ 12 \end{array}$$

Quotient = $\boxed{x^2 + 2x + 1}$

Remainder = $\boxed{12}$

$$\frac{x^3 - x^2 - 5x + 9}{x-3} = \boxed{x^2 + 2x + 1 + \frac{12}{x-3}}$$

5. (6 marks)

Solve the following linear system

$$x + 2y + 3z = 5$$
$$3x + 5y + z = 2$$
$$2x - y - 3z = 7$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 3 & 5 & 1 & 2 \\ 2 & -1 & -3 & 7 \end{array}\right] \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \end{array} \quad \left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & -1 & -8 & -13 \\ 0 & -5 & -9 & -3 \end{array}\right] R_2 \to -R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 1 & 8 & 13 \\ 0 & -5 & -9 & -3 \end{array}\right] R_3 \to R_3 + 5R_2 \quad \left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 1 & 8 & 13 \\ 0 & 0 & 31 & 62 \end{array}\right] R_3 \to \frac{1}{31} R_3$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 1 & 8 & 13 \\ 0 & 0 & 1 & 2 \end{array}\right] \qquad \begin{array}{rl} x + 2y + 3z = 5 & (1) \\ y + 8z = 13 & (2) \\ z = 2 & (3) \end{array}$$

(3) : $\underline{z = 2}$, (2) : $y + 8(2) = 13$, so $y + 16 = 13$

$\underline{y = -3}$

(1) : $x + 2(-3) + 3(2) = 5$

$x - 6 + 6 = 5$, so $\underline{x = 5}$

So the system has <u>one</u> solution : $\boxed{x = 5, \quad y = -3, \quad z = 2}$

6. (5 marks)

Graph the quadratic function $f(x) = -x^2 - 2x + 8$. Identify the x-intercepts (if any), the y-intercept, the vertex and the range of the function.

$$\frac{b^2}{4a} = \frac{(-2)^2}{4(-1)} = \frac{4}{-4} = -1$$

standard form: $f(x) = -x^2 - 2x - 1 + 1 + 8 = -(x^2 + 2x + 1) + 9$

$$= \boxed{-(x+1)^2 + 9}$$

The graph of $y = x^2$ is: (1) shifted left 1 unit, (2) reflected about the x-axis and (3) shifted up 9 units:

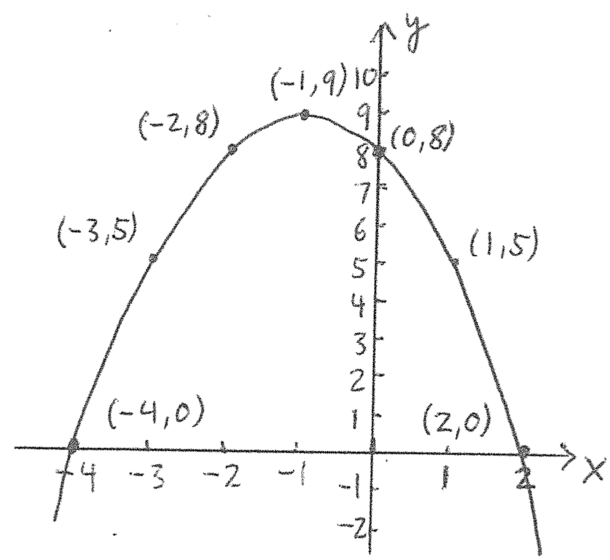
$(-2,4) \longrightarrow (-3,4) \longrightarrow (-3,-4) \longrightarrow (-3,5)$

$(-1,1) \longrightarrow (-2,1) \longrightarrow (-2,-1) \longrightarrow (-2,8)$

$(0,0) \longrightarrow (-1,0) \longrightarrow (-1,0) \longrightarrow (-1,9)$

$(1,1) \longrightarrow (0,1) \longrightarrow (0,-1) \longrightarrow (0,8)$

$(2,4) \longrightarrow (1,4) \longrightarrow (1,-4) \longrightarrow (1,5)$



The vertex is $\boxed{(-1,9)}$

y-intercept: $f(0) = -(0)^2 - 2(0) + 8 = 8$

so $\boxed{(0,8)}$ is the y-intercept.

x-intercepts: $f(x) = 0$, so $-(x+1)^2 + 9 = 0$,

so $(x+1)^2 = 9$, so $x + 1 = 3$ or $x + 1 = -3$

$\underline{x = 2}$ $\qquad$ $\underline{x = -4}$

so the x-intercepts are $\boxed{(2,0)}$ and $\boxed{(-4,0)}$

The range is $\boxed{(-\infty, 9]}$

7. (4 marks)

If $f(x) = \dfrac{1}{x^2-1}$ and $g(x) = \sqrt{x+1}$ then find the composite functions $(f \circ g)(x)$ and $(g \circ f)(x)$

$$(f \circ g)(x) = f(g(x)) = f(\sqrt{x+1}) = \frac{1}{(\sqrt{x+1})^2 - 1}$$

$$= \frac{1}{(x+1)-1} = \boxed{\frac{1}{x}}$$

$$(g \circ f)(x) = g(f(x)) = g\left(\frac{1}{x^2-1}\right) = \sqrt{\left(\frac{1}{x^2-1}\right)+1}$$

$$= \sqrt{\frac{1}{x^2-1} + \frac{x^2-1}{x^2-1}} = \sqrt{\frac{1+(x^2-1)}{x^2-1}}$$

$$= \boxed{\sqrt{\frac{x^2}{x^2-1}}}$$

8. (4 marks)

If $f(x) = \dfrac{2x+1}{x-3}$, find the inverse function $f^{-1}(x)$

$$y = \frac{2x+1}{x-3}$$

inverse: $x = \dfrac{2y+1}{y-3}$, so

$$x(y-3) = 2y+1$$

$$xy - 3x = 2y+1$$

$$xy - 2y = 3x+1$$

$$y(x-2) = 3x+1$$

$$y = \frac{3x+1}{x-2}$$

so $\boxed{f^{-1}(x) = \dfrac{3x+1}{x-2}}$

9. (6 marks)

Find the domain of each of the following functions

(a) $f(x) = \dfrac{\sqrt{x+5}}{x+3}$

Here, $x+5 \geqslant 0$ and $x+3 \neq 0$

$\underline{x \geqslant -5}$ $\underline{x \neq -3}$

So. domain $= \{x \in \mathbb{R} \mid x \geqslant -5 \text{ and } x \neq -3\}$

$= \boxed{[-5,-3) \cup (-3,\infty)}$

(b) $f(x) = \dfrac{x^3}{x^2+5}$

Here $x^2+5 \neq 0$, which is always true since $x^2+5 > x^2 \geqslant 0$, so $\underline{x^2+5>0}$

So domain $= \boxed{(-\infty,\infty)}$

10. (3 marks)

Write the expression $3\log_2 x + \frac{1}{2}\log_2(y+1) - 1$ as a single logarithm.

$$3\log_2 x + \tfrac{1}{2}\log_2(y+1) - 1 = \log_2(x^3) + \log_2((y+1)^{1/2}) - \log_2(2^1)$$

$$= \log_2\left(x^3(y+1)^{1/2}\right) - \log_2(2)$$

$$= \log_2\left(\frac{x^3(y+1)^{1/2}}{2}\right) = \boxed{\log_2\left(\frac{x^3\sqrt{y+1}}{2}\right)}$$

11. (8 marks)

Solve the following equations for x

(a) $3^{3x-4} = 9^{x+1}$

$3^{3x-4} = (3^2)^{x+1}$, so $3^{3x-4} = 3^{2(x+1)}$

$3x-4 = 2(x+1)$

$3x-4 = 2x+2$

$\boxed{x = 6}$

(b) $\log_5(x+9) = \log_5(x+5) + 1$

$\log_5(x+9) - \log_5(x+5) = 1$

$\log_5\left(\frac{x+9}{x+5}\right) = 1$

$\frac{x+9}{x+5} = 5^1 = 5$, so $x+9 = 5(x+5)$

$x+9 = 5x+25$

$-4x = 16$

$\underline{x = -4}$

<u>Check</u>: <u>$x=-4$</u>: $\log_5(-4+9) = \log_5(-4+5)+1$

$\log_5(5) = \log_5(1)+1$

$1 = 0+1$, $1=1$ ✓

so $\boxed{x = -4}$ is a solution.

12. (4 marks)

The current i (measured in A) in an electric circuit is given by $i = 20\,(1 - e^{-75t})$ where t is the time (measured in s). How long does it take for the current to reach 19.5 A ?

Find t for which $i = 19.5$:

$19.5 = 20(1 - e^{-75t})$, so $\frac{19.5}{20} = 1 - e^{-75t}$, so

$e^{-75t} = 1 - \frac{19.5}{20} = \frac{1}{40}$, so $-75t = \ln\left(\frac{1}{40}\right)$

$t = \frac{\ln\left(\frac{1}{40}\right)}{-75} \cong 0.049$, so it takes 0.049 s,

i.e. $\boxed{49 \text{ ms}}$ for the current to reach 19.5 A.

13. (6 marks)

Find the **exact** values of each of the following (**Important**: no marks will be given for decimal approximations)

(a) $\cos\left(\frac{5\pi}{4}\right)$ $\theta = \frac{5\pi}{4}\left(\frac{180^\circ}{\pi}\right) = 225^\circ$ is in Quadrant 3 :

$\theta_R = \theta - 180^\circ = 225^\circ - 180^\circ = 45^\circ$

$\cos\left(\frac{5\pi}{4}\right) = \cos(225^\circ) = -\cos(45^\circ) = \boxed{-\frac{1}{\sqrt{2}}}$

(b) $\csc(300^\circ)$ $\theta = 300^\circ$ is in Quadrant 4 :

$\theta_R = 360^\circ - \theta = 360^\circ - 300^\circ = 60^\circ$

$\csc(300^\circ) = -\csc(60^\circ) = \boxed{-\frac{2}{\sqrt{3}}}$

(c) $\tan\left(\frac{17\pi}{6}\right)$ $\theta = \frac{17\pi}{6}\left(\frac{180^\circ}{\pi}\right) = 510^\circ$, is co-terminal with 150° (Quadrant 2):

$\theta_R = 180^\circ - 150^\circ = 30^\circ$

$\tan\left(\frac{17\pi}{6}\right) = \tan(510^\circ) = \tan(150^\circ) = -\tan(30^\circ) = \boxed{-\frac{1}{\sqrt{3}}}$

14. (3 marks)

Verify the trigonometric identity $\sec\theta - \sin\theta\,\tan\theta = \cos\theta$

Show all your steps.

$$\sec\theta - \sin\theta \tan\theta = \cos\theta$$

$$\frac{1}{\cos\theta} - \sin\theta\left(\frac{\sin\theta}{\cos\theta}\right) = \cos\theta$$

$$\frac{1}{\cos\theta} - \frac{\sin^2\theta}{\cos\theta} = \cos\theta$$

$$\frac{1-\sin^2\theta}{\cos\theta} = \cos\theta$$

$\sin^2\theta + \cos^2\theta = 1$, so $\dfrac{\cos^2\theta}{\cos\theta} = \cos\theta$, so $\cos\theta = \cos\theta$ ✓

15. (4 marks)

Solve the trigonometric equation $2\sin x\cos x + \sqrt{3}\sin x = 0$ for x, where $0 \le x < 2\pi$. Give exact solutions.

$$\sin x\,(2\cos x + \sqrt{3}) = 0$$

so either $\underline{\sin x = 0}$ or $\underline{2\cos x + \sqrt{3} = 0}$

If $\sin x = 0$, then $\underline{x = 0}$ or $\underline{x = \pi}$

If $2\cos x + \sqrt{3} = 0$, then $2\cos x = -\sqrt{3}$, so $\underline{\cos x = -\frac{\sqrt{3}}{2}}$

$\cos x < 0$, so x is in Quadrants 2 and 3.

$\cos\theta_R = |\cos x| = \left|-\frac{\sqrt{3}}{2}\right| = \frac{\sqrt{3}}{2}$, so $\underline{\theta_R = \frac{\pi}{6}}$

In Quadrant 2: $\theta_R = \pi - x$, so $x = \pi - \theta_R = \pi - \frac{\pi}{6} = \underline{\frac{5\pi}{6}}$

In Quadrant 3: $\theta_R = x - \pi$, so $x = \theta_R + \pi = \frac{\pi}{6} + \pi = \underline{\frac{7\pi}{6}}$

Solutions: $\boxed{x = 0}$, $\boxed{x = \frac{5\pi}{6}}$, $\boxed{x = \pi}$, $\boxed{x = \frac{7\pi}{6}}$

16. (5 marks)

Graph one cycle of the function $y = -2\sin\left(2x + \frac{\pi}{2}\right)$

Identify the amplitude, period, phase shift and vertical shift of the function.

$y = A\sin(\omega x + \phi) + K$, where $A = -2$, $\omega = 2$,

$\phi = \frac{\pi}{2}$, $K = 0$

amplitude $= |A| = |-2| = \boxed{2}$, period $= \frac{2\pi}{\omega} = \frac{2\pi}{2} = \boxed{\pi}$,

phase shift $= -\frac{\phi}{\omega} = -\frac{\left(\frac{\pi}{2}\right)}{2} = \boxed{-\frac{\pi}{4}}$, vertical shift $= K = \boxed{0}$

The graph of $y = \sin x$ is (1) shifted left by $\frac{\pi}{2}$ units. (2) compressed horizontally by a factor of 2 and (3) stretched vertically by a factor of 2 and reflected about the x-axis.

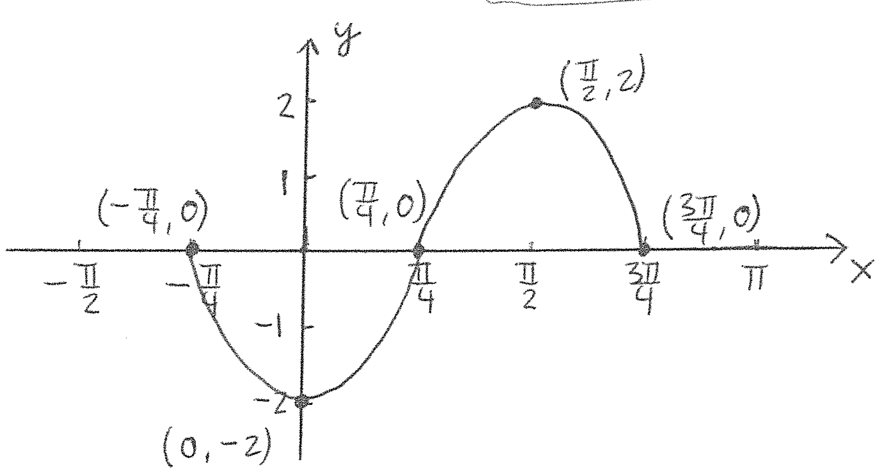
$(0,0) \to (-\frac{\pi}{2}, 0) \to (-\frac{\pi}{4}, 0) \to (-\frac{\pi}{4}, 0)$

$(\frac{\pi}{2}, 1) \to (0, 1) \to (0, 1) \to (0, -2)$

$(\pi, 0) \to (\frac{\pi}{2}, 0) \to (\frac{\pi}{4}, 0) \to (\frac{\pi}{4}, 0)$

$(\frac{3\pi}{2}, -1) \to (\pi, -1) \to (\frac{\pi}{2}, -1) \to (\frac{\pi}{2}, 2)$

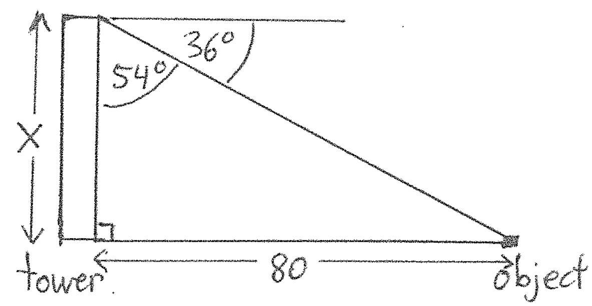
$(2\pi, 0) \to (\frac{3\pi}{2}, 0) \to (\frac{3\pi}{4}, 0) \to (\frac{3\pi}{4}, 0)$



17. (4 marks)

From the top of an observation tower, the angle of depression to an object on the ground is 36°. If the object is 80 m from the base of the tower, what is the height of the tower?

Let x be the height of the tower (in m).



$\tan(54°) = \frac{80}{x}$, so

$x = \frac{80}{\tan(54°)} = \boxed{58.1}$

The tower is $\boxed{58 \text{ m}}$ high.

18. (6 marks)

Perform the following complex number operations. Write each answer in the form $a+bj$.

(a) $(1+2j)\ (3-4j)$

$(1+2j)(3-4j) = 3 - 4j + 6j - 8j^2 = 3 + 2j - 8(-1)$

$= \boxed{11 + 2j}$

(b) $\frac{5-2j}{1+3j}$

$\frac{5-2j}{1+3j} = \frac{5-2j}{1+3j} \cdot \frac{1-3j}{1-3j} = \frac{5-15j-2j+6j^2}{1-9j^2} = \frac{5-17j+6(-1)}{1-9(-1)}$

$= \frac{-1-17j}{1+9} = \boxed{-\frac{1}{10} - \frac{17}{10}j}$

19. (4 marks)

Find all complex number solutions of the equation $z^3 + 5z^2 + 3z + 15 = 0$

Hint: factor by grouping

$$z^3 + 5z^2 + 3z + 15 = 0$$
$$z^2(z+5) + 3(z+5) = 0$$
$$(z^2+3)(z+5) = 0$$

either $z^2 + 3 = 0$ or $z+5=0$

$z^2 = -3$ $\quad$ $\underline{z = -5}$

$z = \pm\sqrt{-3} = \underline{\pm\sqrt{3}j}$

Solutions: $\boxed{z=-5}$, $\boxed{z=\sqrt{3}j}$, $\boxed{z=-\sqrt{3}j}$

20. (3 marks)

Find the polar form of the complex number $z = -4\sqrt{3} + 4j$

$z = a + bj$, where $a = -4\sqrt{3}$ and $b = 4$

$$r = \sqrt{a^2+b^2} = \sqrt{(-4\sqrt{3})^2 + (4)^2} = \sqrt{48+16} = \sqrt{64} = 8$$

$$\tan\theta = \frac{b}{a} = \frac{4}{-4\sqrt{3}} = -\frac{1}{\sqrt{3}}$$

Also, θ is clearly in Quadrant 2.

$\tan\theta_R = |\tan\theta| = \left|-\frac{1}{\sqrt{3}}\right| = \frac{1}{\sqrt{3}}$, so $\theta_R = \frac{\pi}{6}$

Quadrant 2: $\theta_R = \pi - \theta$, so $\theta = \pi - \theta_R = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

So $z = r\,\text{cis}\,\theta = \boxed{8\,\text{cis}\left(\frac{5\pi}{6}\right)}$ is the polar form.