DAWSON COLLEGE

Mathematics Department

Final Examination

201-BZF-05 (Calculus III), Section 00001

Fall 2021

Date: Tuesday December 21, 2021, Time: 9:30 am – 12:30 pm

Examiner: S. Shahabi.

Name: ______________________ **ID:** ______________________

- Print your name and student ID number in the space provided above;
- All questions are to be answered directly on the examination paper in the space provided. If you need more space for your answer, use the back of the page;
- No book, notes, graphing/programmable calculator or cellphones are permitted;
- You are only permitted to use the **Sharp EL-531**** calculator;
- Show all your work and justify all your answers;
- This examination booklet must be returned intact;

1)	9)
2)	10)
3)	11)
4)	12)
5)	13)
6)	14)
7)	15)
8)	16)

THIS EXAMINATION BOOKLET CONTAINS 9 PAGES (INCLUDING THIS COVER PAGE), AND 16 QUESTIONS.

$$\left(\frac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots \quad -1 < t < 1 \right)$$

(1) **[5 pts.]** Expand $f(x) = \dfrac{1}{x^3 - 3x^2 + 3x}$ in powers of $(x-1)$, and determine the interval of convergence of the power series. **Hint:** the denominator is equal to $1 - (1-x)^3$.

$$f(x) = \frac{1}{1-(1-x)^3} = 1 + (1-x)^3 + (1-x)^6 + (1-x)^9 + \cdots$$
$$= 1 - (x-1)^3 + (x-1)^6 - (x-1)^9 + \cdots$$

It is convergent $\iff -1 < (1-x)^3 < 1$

$\iff -1 < 1-x < 1$

$\iff -1 < x-1 < 1$

Hence : $\begin{cases} R = 1, & \text{radius} \\ I = \left]0, 2\right[, & \text{interval} \end{cases}$

(2) **[5 pts.]** Find the *exact value* of $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{3^n \sqrt{3}(2n+1)}$. **Recall:** $\arctan(x) = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} x^{2n-1}$. $(*)$

$$L = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n \cdot \sqrt{3}(2n+1)} \overset{m=n+1}{=\!=} \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{3^{m-1} \cdot 3^{\frac{1}{2}} (2m-1)}$$

$$= \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{2m-1} \cdot \left(\frac{1}{\sqrt{3}}\right)^{2m-1} \overset{(*)}{=} \arctan\left(\frac{1}{\sqrt{3}}\right) = \boxed{\frac{\pi}{6}}$$

[Also note:
$$\arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{2} - \arctan(\sqrt{3})$$
using:
$$\left(\arctan(x) + \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2} \quad x > 0 \right)$$
$$= \frac{\pi}{2} - \frac{\pi}{3} = \pi/6$$
]

(3) **[4 pts.]** If $-1 \le a < 0$, show $\left|\sum_{n=1}^{\infty} \frac{a^n}{n^2+1} - \sum_{n=1}^{N} \frac{a^n}{n^2+1}\right| < \frac{1}{(N+1)^2}$. **Hint:** a^n is $\begin{cases} \text{positive}, & \text{if } n = 2k, \\ \text{negative}, & \text{if } n = 2k+1. \end{cases}$

Recall: for alternating series $S = \sum_{1}^{\infty} (-1)^n b_n$, we have $|S - S_N| < b_{N+1}$.

Applying this inequality, we get

$$\left| \sum_{1}^{\infty} \frac{(-1)^n (-a)^n}{n^2+1} - \sum_{1}^{N} \frac{(-1)^n(-a)^n}{n^2+1} \right| < \frac{(-a)^{N+1}}{(N+1)^2+1} \le \frac{1}{(N+1)^2+1}$$

$(0 < -a \le 1)$

$$< \frac{1}{(N+1)^2}.$$

(4) **[6 pts.]** Find the arc-length parametrization of the curve $\vec{\mathbf{r}}(t) = \left\langle \frac{2t}{t^2+1}, \pi, \frac{t^2-1}{t^2+1} \right\rangle$, $t \ge 0$, *starting at zero.*

$$s(t) = \int_0^t \|\vec{r}\,'(u)\|\, du \qquad \vec{r}\,'(u) = \left\langle \frac{2(1-u^2)}{(1+u^2)^2}, 0, \frac{4u}{(1+u^2)^2} \right\rangle$$

$$= \int_0^t \frac{2}{1+u^2}\, du \qquad \|\vec{r}\,'(u)\| = \frac{2}{(1+u^2)}$$

$$= 2\left[\arctan(u)\right]_0^t$$

$$\left(\sin(2\alpha) = \frac{2\tan(\alpha)}{1+\tan^2(\alpha)} \quad \cos(2\alpha) = \frac{1-\tan^2(\alpha)}{1+\tan^2(\alpha)} \right)$$

$$= 2\arctan(t) \quad (t \ge 0)$$

$\longrightarrow$ $\arctan(t) = \frac{1}{2}s \longrightarrow t = \tan(\frac{1}{2}s)$

$\longrightarrow$ $\vec{r} = \vec{r}(s) = \left\langle \frac{2\tan(\frac{1}{2}s)}{1+\tan^2(\frac{1}{2}s)}, \pi, \frac{\tan^2(\frac{1}{2}s)-1}{1+\tan^2(\frac{1}{2}s)} \right\rangle$

$= \langle \sin(s), \pi, -\cos(s) \rangle$. (s: the new parameter)

(RmK: This is a circle (of radius 1) in a plane parallel to the xz-plane.)

(5) **[6 pts.]** Compute the curvature of $\vec{\mathbf{r}}(t) = \left\langle \frac{2t}{t^2+1}, \pi, \frac{t^2-1}{t^2+1} \right\rangle$ at $t = 1$. **Recall:** $\kappa = \frac{\|\vec{\mathbf{r}}\,'(t) \times \vec{\mathbf{r}}\,''(t)\|}{\|\vec{\mathbf{r}}\,'(t)\|^3}$.

$$\vec{r}\,'(t) = \left\langle \frac{2-2t^2}{(1+t^2)^2}, 0, \frac{4t}{(1+t^2)^2} \right\rangle$$

$$\vec{r}\,''(t) = \left\langle \frac{4t^3-12t}{(1+t^2)^3}, 0, \frac{4-12t^2}{(1+t^2)^3} \right\rangle$$

Specializing to $t=1$:

$$\vec{r}\,'(1) = \underbrace{\langle 0,0,1\rangle}_{=\vec{K}}, \quad \vec{r}\,''(1) = \underbrace{\langle -1,0,-1\rangle}_{=-\vec{i}-\vec{K}}$$

$$\Rightarrow \kappa(1) = \frac{\|\langle 0,0,1\rangle \times \langle -1,0,-1\rangle\|}{\|\langle 0,0,1\rangle\|^3}$$

$$= \frac{\|\overbrace{\langle 0,-1,0\rangle}^{\vec{j}}\|}{(1)^3} = \frac{1}{1} = \boxed{1}$$

$$\left(\vec{K} \times (-\vec{i}-\vec{K}) = -\vec{K}\times\vec{i} - \underbrace{\vec{K}\times\vec{K}}_{\vec{0}} = \vec{i}\times\vec{K} = -\vec{j} \right)$$

(Rmk: An expected answer; due to the previous problem, this is a circle of radius 1, and for a circle of radius R, $\kappa = 1/R$.)

(6) **[4 pts.]** Show that if $a \neq b$, then the tangent lines to the curve $\vec{\mathbf{r}}(t) = \langle t^3, t^6, t^9 \rangle$ at $t = a$ and at $t = b$ *cannot* be parallel.

They would if $\Leftrightarrow \vec{r}\,'(a) \parallel \vec{r}\,'(b)$

$$\Leftrightarrow \langle 3a^2, 6a^5, 9a^8\rangle \parallel \langle 3b^2, 6b^5, 9b^8\rangle$$

$$\Leftrightarrow \frac{a^2}{b^2} = \frac{a^5}{b^5} = \frac{a^8}{b^8}$$ which is impossible unless $a=b$.

(Note: we may assume $a, b \neq 0$)

(7) **[5 pts.]** Find $\lim_{(x,y)\to(0,0)} \underbrace{\frac{x^3y^3\sqrt{|\sin(x^2+y^2)|}}{x^6+y^6}}_{= f(x,y)}$ or show it does not exist. **Hint:** $-1 \le \frac{2ab}{a^2+b^2} \le 1$. $(a,b)\neq(0,0)$

Exploiting the given inequality, we have $(a=x^3, b=y^3)$

$(x,y)\neq(0,0)$

$$-\frac{1}{2} \le \frac{x^3y^3}{x^6+y^6} \le \frac{1}{2}$$

. Now by multiplying by $\sqrt{|\sin(x^2+y^2)|}$, we get (in some neighbourhood of the origin)

$$-\frac{1}{2}\sqrt{|\sin(x^2+y^2)|} \le f(x,y) \le \frac{1}{2}\sqrt{|\sin(x^2+y^2)|}$$

Sending (x,y) to $(0,0)$, and applying Sandwitch/Squeeze theorem, we establish that

$\lim_{(x,y)\to(0,0)} f(x,y)$ exists and is equal to zero.

(8) **[5 pts.]** Find an equation of the tangent plane to the surface $\underbrace{ye^{x/y} - z}_{f(x,y,z)} = 0$ at the point $P(1,1,e)$.

$$\frac{\partial f}{\partial x}(P)\cdot(x-1) + \frac{\partial f}{\partial y}(P)\cdot(y-1) + \frac{\partial f}{\partial z}(P)\cdot(z-e) = 0$$

or $e\cdot(x-1) + 0\cdot(y-1) + (-1)(z-e) = 0$

or $ex - e - z + e = 0$

$$\left(\begin{aligned} \frac{\partial f}{\partial x} &= e^{x/y} \\ \frac{\partial f}{\partial y} &= e^{x/y} - \frac{x}{y}e^{x/y} \\ \frac{\partial f}{\partial z} &= -1 \end{aligned}\right)$$

$\boxed{ex - z = 0}$

(9) **[5 pts.]** If $c_1^2+c_2^2+\cdots+c_n^2=c$ (they are all constants), show that $z=e^{c_1x_1+c_2x_2+\cdots+c_nx_n}$ satisfies the (partial differential) equation

$$\frac{\partial^2 z}{\partial x_1^2}+\frac{\partial^2 z}{\partial x_2^2}+\cdots+\frac{\partial^2 z}{\partial x_n^2}=cz.$$

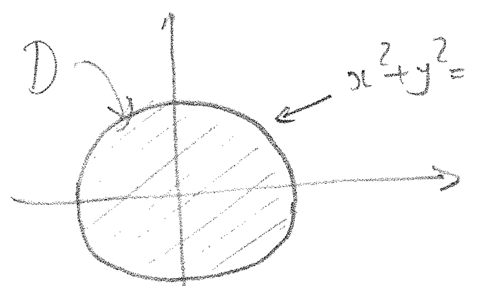
$$\frac{\partial z}{\partial x_i} = c_i \cdot e^{c_1x_1+\cdots+c_nx_n} = c_i \cdot z \quad (\text{for } i=1,2,\dots,n)$$

$$\Rightarrow \frac{\partial^2 z}{\partial x_i^2} = c_i^2 \cdot z \quad (1 \le i \le n)$$

$$\Rightarrow \sum_{i=1}^{n} \frac{\partial^2 z}{\partial x_i^2} = \sum_{i=1}^{n} c_i^2 \cdot z = \left(\sum_{i=1}^{n} c_i^2\right) z = c \cdot z \checkmark$$

as desired!

(10) **[6 pts.]** Find the extreme values of $f(x,y)=1+e^{-x^2-y^2}$, over the unit disk $\{(x,y): x^2+y^2\le 1\}$.



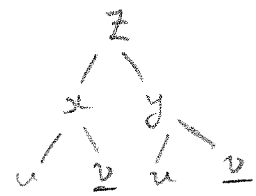
- Over the boundary

$$f(x,y) = 1+e^{-(x^2+y^2)} = 1+e^{-1} = \frac{e+1}{e}$$

($\therefore$ f is constant on the boundary)

- Critical point(s) (inside) $\begin{cases} \frac{\partial f}{\partial x}=0 \\ \frac{\partial f}{\partial y}=0 \end{cases} \rightarrow \begin{cases} -2x \cdot e^{-x^2-y^2}=0 \\ -2y \cdot e^{-x^2-y^2}=0 \end{cases} \rightarrow \boxed{x=y=0}$

Since $f(0,0)=2 > \frac{e+1}{e} \longrightarrow \begin{cases} \mathrm{Max}_D(f) = 2 \\ \mathrm{Min}_D(f) = \frac{e+1}{e} \end{cases}$



(11) **[5 pts.]** Find $\dfrac{\partial z}{\partial v}$ if $z = x^2 + y^2 + \sin(xy)$, $x = \sin(u - v)$, and $y = \cos(u + v)$.

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

$$= (2x + y\cos(xy)) \cdot (-\cos(u-v)) + (2y + x\cos(xy)) \cdot (-\sin(u+v))$$

(12) **[6 pts.]** Compute the integral $I = \displaystyle\iint_R \frac{1}{1+x+y}\, dA$ over the rectangle $R = [1,3] \times [1,2]$.

$$I = \int_1^2 \left(\int_1^3 \frac{1}{1+x+y}\, dx \right) dy = \int_1^2 \Big[\ln(1+x+y)\Big]_1^3 dy$$

$$\left(\int \ln(u)\, du = u(\ln(u) - 1) + C \right)$$

$$= \int_1^2 (\ln(y+4) - \ln(y+2))\, dy$$

$$= \Big[(y+4)(\ln(y+4) - 1) - (y+2)(\ln(y+2) - 1)\Big]_1^2$$

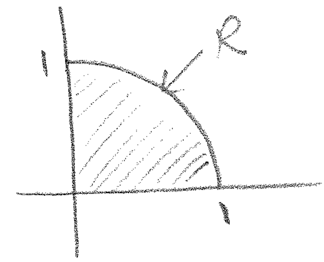
$$= \{6(\ln(6) - 1) - 4(\ln(4) - 1)\} - \{5\cdot(\ln(5) - 1) - 3\cdot(\ln(3) - 1)\}$$

$$= 6\ln(6) - 4\ln(4) - 5\ln(5) + 3\ln(3) - 6 + 4 + 5 - 3$$

$$= \ln\left(\frac{6^6 \cdot 3^3}{5^5 \cdot 4^4}\right) = \boxed{\ln\left(\frac{3^9}{4 \cdot 5^5}\right)}$$

I

(13) **[6 pts.]** Compute the integral $\iint_D xy\sqrt{x^2+y^2}\,dA$ over the set $D = \{(x,y)\,|\,x^2+y^2 \le 1,\ x \ge 0,\ y \ge 0\}$.



$$I = \int_0^{\pi/2}\left(\int_0^1 r\cos(\theta)\, r\sin(\theta)\sqrt{r^2}\cdot r\, dr\right) d\theta$$

$$= \int_0^{\pi/2} \sin(\theta)\cos(\theta)\left[\tfrac{1}{5}r^5\right]_0^1 d\theta$$

$$= \frac{1}{10}\int_0^{\pi/2} \sin(2\theta)\, d\theta$$

$$= \frac{1}{10}\left(-\frac{1}{2}\cos(2\theta)\right)_0^{\pi/2} = -\frac{1}{20}\left(\cos(\pi) - \cos(0)\right)$$

$$= \boxed{\frac{1}{10}}$$

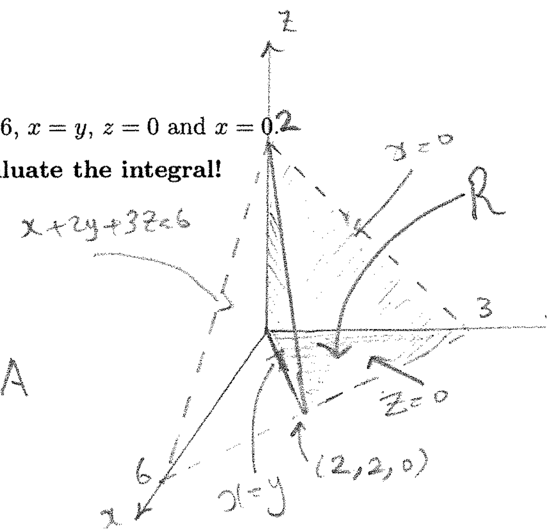
(14) **[8 pts.]** Consider the solid S (in $\mathbb{R}^3$) bounded by the planes $x+2y+3z=6$, $x=y$, $z=0$ and $x=0$.2

(i) Setup a *double-iterated* integral which gives the volume of S; **Don't evaluate the integral!**

$z = f(x,y) = \frac{1}{3}(6 - x - 2y)$

$R:\ 0 \le x \le 2,\quad x \le y \le \frac{1}{2}(6-x)$

$$V = \int_0^2\left(\int_x^{\frac{1}{2}(6-x)} \frac{1}{3}(6 - x - 2y)\, dy\right) dx = \iint_R f\, dA$$



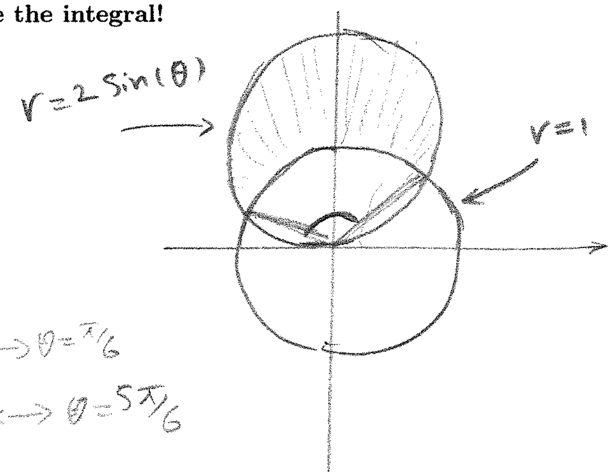
(ii) Setup a *triple-iterated* integral which gives the same volume; **Don't evaluate the integral!**

$$V = \iiint_D 1\, dV = \int_0^2\left(\int_x^{\frac{1}{2}(6-x)}\left(\int_0^{\frac{1}{3}(6-x-2y)} 1\, dz\right) dy\right) dx$$

$r = 2\sin(\theta)$ $r=1$

(15) **[5 pts.]** Setup a double integral **in polar coordinates** which gives the area of the region inside the circle $x^2 + (y-1)^2 = 1$ but outside the circle $x^2 + y^2 = 1$; **Don't evaluate the integral!**

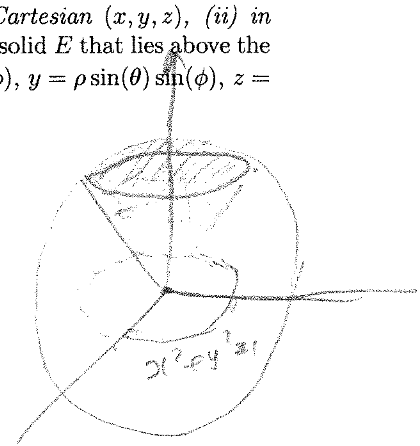
$$A = \int_{\pi/6}^{5\pi/6} \left(\int_{1}^{2\sin(\theta)} 1 \cdot r \cdot dr \right) d\theta$$



$$\begin{cases} x^2+y^2=1 \\ x^2+(y-1)^2=1 \end{cases} \longrightarrow y=\tfrac{1}{2} \rightarrow x = \pm\tfrac{\sqrt{3}}{2} \begin{cases} (\tfrac{\sqrt{3}}{2}, \tfrac{1}{2}) \longleftrightarrow \theta = \pi/6 \\ (-\tfrac{\sqrt{3}}{2}, \tfrac{1}{2}) \longleftrightarrow \theta = 5\pi/6 \end{cases}$$

(16) **[12 pts.]** Setup, **but do not evaluate,** *three triple-iterated integrals: (i) in Cartesian (x, y, z), (ii) in Cylindrical (r, θ, z) and (iii) in Spherical (ρ, θ, ϕ) coordinates,* for the volume of the solid E that lies above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2 = 2$. **Recall:** $x = \rho\cos(\theta)\sin(\phi)$, $y = \rho\sin(\theta)\sin(\phi)$, $z = \rho\cos(\phi)$, and $dV = \rho^2\sin(\phi)d\rho\, d\theta\, d\phi$.

(i) $V = \displaystyle\int_{-1}^{1} \left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left(\int_{\sqrt{x^2+y^2}}^{\sqrt{2-(x^2+y^2)}} 1 \cdot dz \right) dy \right) dx$



(ii) $V = \displaystyle\int_{0}^{2\pi} \left(\int_{0}^{1} \left(\int_{r}^{\sqrt{2-r^2}} 1 \cdot r \cdot dz \right) dr \right) d\theta$

(iii) $V = \displaystyle\int_{\pi/4}^{\pi/2} \left(\int_{0}^{2\pi} \left(\int_{0}^{\sqrt{2}} \rho^2 \sin(\phi)\, d\rho \right) d\theta \right) d\phi$