DAWSON COLLEGE

Mathematics Department

Final Examination

201-BZF-05 (Calculus III), Sections 1,2.

Date: Tuesday May 24, 2022, Time: 9:30 – 12:30

Examiners: R. Fournier and S. Shahabi.

Student's Name: ____________________ **ID:** ____________________

- Print your full name and student ID number in the space provided above;
- All questions are to be answered directly on the examination paper in the space provided. If you need more space for your answer, use the back of the page;
- No book, notes, or electronic devices are permitted. You are only permitted to use the **Sharp EL-531**** calculator;
- Show all your work and justify all your answers;
- This examination booklet must be returned intact;

Question	Marks	student's Score
1	6	
2	5	
3	6	
4	5	
5	5	
6	5	
7	5	
8	5	
9	6	
10	6	
11	6	
12	6	
13	6	
14	6	
15	5	
16	5	
17	6	
18	6	
Total	100	

THIS EXAMINATION BOOKLET MUST BE RETURNED INTACT. IT CONTAINS 10 PAGES (INCLUDING THIS COVER PAGE), AND 18 QUESTIONS.

(1) **[6 pts]** Find the interval of convergence as well as a closed form (i.e., the sum!) for the power series $\sum_{n=1}^{\infty} n^2(x-3)^{4n}$. **(Hint:** it might help if you consider the substitution $t=(x-3)^4$)

$$\sum_{0}^{\infty} t^n = \frac{1}{1-t} \overset{1}{\underset{\text{diff.}}{\Longrightarrow}} \sum_{0}^{\infty} n t^{n-1} = \frac{1}{(1-t)^2} \overset{1}{\Longrightarrow} \sum_{n=0}^{\infty} n t^n = \frac{t}{(1-t)^2}$$

$$\overset{1}{\underset{\text{diff.}}{\Longrightarrow}} \sum_{n=1}^{\infty} n^2 t^{n-1} = \frac{1+t}{(1-t)^3} \overset{1}{\Longrightarrow} \sum_{1}^{\infty} n^2 t^n = \frac{t+t^2}{(1-t)^3}$$

$$\overset{1}{\underset{t=(x-3)^4}{\Longrightarrow}} \sum_{1}^{\infty} n^2 (x-3)^{4n} = \frac{(x-3)^4 + (x-3)^8}{(1-(x-3)^4)^3}$$

$R = 1$, $I = (2, 4)$ 1

(2) **[5 pts]** Find the length of the curve with parametric equations $x=3t^2$, $y=2t^3$, & $z=1$, if $0 \le t \le 2$.

$$l \overset{1}{=} \int_0^2 \sqrt{(6t)^2 + (6t^2)^2 + (0)^2}\, dt \overset{1}{=} 6\int_0^2 \sqrt{t^2+t^4}\, dt$$

$$\overset{1}{=} 6 \int_0^2 t(1+t^2)^{1/2}\, dt \overset{1}{=} 2(1+t^2)^{3/2}\Big|_0^2$$

$$= 2\sqrt{125} - 2 = 2(5\sqrt{5}-1) \quad (\approx 20.36)$$

1

(3) **[6 pts]** Find a power series representation for $f(x) = -\ln(4-x)$, and then give its associated radius of convergence.

Solution (I)

center "0": $-\ln(4-x) = -\ln 4 - \ln(1 - \frac{1}{4}x)$ 2

2 $= -\ln 4 + \sum_{n=1}^{\infty} \frac{1}{4^n n} x^n$ $R = 4$ 2

Solution (II)

center "3": $-\ln(1-(x-3)) = \sum_{n=1}^{\infty} \frac{1}{n}(x-3)^n$ $R=1$ 2

4

(Both solutions are using $-\ln(1-t) = \sum_{n\geq 1} \frac{1}{n} t^n$ with $R = 1$)

(4) **[5 pts]** Show that the graph of $f(x) = \sqrt{\pi^2 - x^2}$ has *constant curvature* $\kappa = \dfrac{|f''(x)|}{(1+|f'(x)|^2)^{3/2}}$, if $|x| < \pi$.

$f'(x) = \dfrac{-x}{\sqrt{\pi^2 - x^2}}$ 1.5 $f''(x) = \dfrac{-\pi^2}{(\pi^2 - x^2)^{3/2}}$ 1.5

$$\Rightarrow \kappa = \frac{\dfrac{\pi^2}{(\pi^2-x^2)^{3/2}}}{\left(\dfrac{\pi^2}{\pi^2-x^2}\right)^{3/2}} = \frac{\pi^2}{\pi^3} = \frac{1}{\pi} = 1/R$$

2

(It is a (semi)circle whose curvature is $= 1/\text{radius}$)

(5) **[5 pts]** Parametrize the curve $\vec{r}(t) = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$, $t \geq 1$, *with respect to arc-length.* **What can you say about this curve?**

$$\vec{r}\,'(t) = \left(\frac{-4t}{(1+t^2)^2}, \frac{2(1-t^2)}{(1+t^2)^2}\right) \Rightarrow \|\vec{r}\,'(t)\| = \frac{2}{1+t^2} \quad 2.5$$

$$s = \int_1^t \frac{2}{1+u^2}\,du = 2\arctan(t) - \pi/2 \quad 1$$

$$\rightarrow \arctan(t) = \frac{1}{2}(s + \pi/2) \longrightarrow t = \tan\left(\frac{1}{2}s + \pi/4\right) \quad 1$$

$$\Rightarrow \vec{r} = \vec{r}(s) = \left(\frac{1 - \tan^2(\frac{1}{2}s + \pi/4)}{1 + \tan^2(\frac{1}{2}s + \pi/4)}, \quad 0.5 \quad \frac{2\tan(\frac{1}{2}s + \pi/4)}{1 + \tan^2(\frac{1}{2}s + \pi/4)}\right)$$

$$= \left(\cos(s + \pi/2), \sin(s + \pi/2)\right) = \left(-\sin(s), \cos(s)\right)$$

(a circle!)

(6) **[5 pts]** For a smooth vector function $\vec{r}(t)$, prove that $\mathbf{T}(t) = \frac{1}{\|\vec{r}\,'(t)\|}\vec{r}\,'(t)$ and $\mathbf{T}'(t)$ are perpendicular.

Since $\|T(t)\| = 1$ (constant), the result follows 1

from: <u>Thm</u>: If $\|\vec{r}\| = c \rightarrow \vec{r}(t) \perp \vec{r}\,'(t)$ 4

[<u>Proof</u>: $\|\vec{r}\| = c \Rightarrow \vec{r}(t) \cdot \vec{r}(t) = c^2 \Rightarrow \frac{d}{dt}(\vec{r}(t) \cdot \vec{r}(t)) = 0$

$$\Rightarrow \vec{r}\,'(t) \cdot \vec{r}(t) + \vec{r}(t) \cdot \vec{r}\,'(t) = 0$$

$$\Rightarrow \cancel{2}\,\vec{r}(t) \cdot \vec{r}\,'(t) = 0 \Rightarrow \vec{r}(t) \perp \vec{r}\,'(t).]$$

(7) **[5 pts]** Answer "True or False" with a short justification: if $f(x, y)$ has partial derivatives at $O(0.0)$, then f must be continuous there.

False: (counter)-example. $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$ 1

One easily verifies:

- $\dfrac{\partial f}{\partial x}(0,0) = 0$ (exists) 1
- $\dfrac{\partial f}{\partial y}(0,0) = 0$ (") 1
- $\lim\limits_{\substack{(x,y)\to(0,0)\\ x=0}} f(x,y) = 0$ but $\lim\limits_{\substack{(x,y)\to(0,0)\\ x=y}} f(x,y) = \dfrac{1}{2}$ $\Rightarrow$ f is not cont. at $(0,0)$. 2

(8) **[5 pts]** If $z = \sin(x + \sin(t))$, then verify that $\dfrac{\partial z}{\partial x} \cdot \dfrac{\partial^2 z}{\partial x \partial t} = \dfrac{\partial z}{\partial t} \cdot \dfrac{\partial^2 z}{\partial x^2}$.

1 $\dfrac{\partial z}{\partial x} = \cos(x + \sin(t)) \cdot 1$ 1 $\longrightarrow$ $\dfrac{\partial^2 z}{\partial x \partial t} = -\sin(x + \sin(t)) \cdot \cos(t)$ 1

$\Rightarrow$ L.H.S $= \underbrace{-\cos(t)\cos(x+\sin(t)) \cdot \sin(x+\sin(t))}$ 0.5

2 $\dfrac{\partial z}{\partial t} = \cos(x+\sin(t)) \cdot \cos(t)$ 1 & $\dfrac{\partial^2 z}{\partial x^2} = -\sin(x+\sin(t)) \cdot 1$ 1

$\Rightarrow$ R.H.S $= \underbrace{-\cos(t)\cos(x+\sin(t)) \cdot \sin(x+\sin(t))}$ 0.5

1 + 2 $\Longrightarrow$ The desired equality.

(9) **[6 pts]** Show that $f(x,y) = \begin{cases} \frac{xy\ln(1+|xy|)}{x^2+y^2}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0), \end{cases}$ is **everywhere** continuous.

• The statement is obvious if $(x,y) \neq (0,0)$ 1

• At $(0,0)$: $0 \leq |f(x,y) - \underbrace{f(0,0)}_{0}| = \dfrac{|xy|\,|\ln(1+|xy|)|}{x^2+y^2}$

$\left(-\frac{1}{2} \leq \frac{ab}{a^2+b^2} \leq \frac{1}{2}\right)$ $\leq \frac{1}{2}\ln(1+|xy|)$ 3

Since $\ln(1+|xy|) \longrightarrow 0$ as $(x,y) \longrightarrow (0,0)$, the Squeeze Thm. implies that $f(x,y) \longrightarrow 0$ as $(x,y) \longrightarrow (0,0)$, which is exactly what we wanted. 2

(10) **[6 pts]** Assuming a, b, c are constant, show that the absolute maximum of $f(x,y) = \dfrac{(ax+by+c)^2}{x^2+y^2+1}$ is equal to $a^2+b^2+c^2$.

let $\vec{u} = (a,b,c)$, $\vec{v} = (x,y,1)$. By Cauchy-Swartz inequ. we have $|\vec{u}\cdot\vec{v}| \leq \|\vec{u}\|\,\|\vec{v}\|$

or $|ax+by+c| \leq (a^2+b^2+c^2)^{1/2} \cdot (x^2+y^2+1)^{1/2}$ 3

which can be transformed into $f(x,y) \leq a^2+b^2+c^2$. 2

N.B. The condition for the equality in Cauchy-Swartz is

$\frac{x}{a} = \frac{y}{b} = \frac{1}{c} \iff x = a/c,\ y = b/c$ 1

And $f(\frac{a}{c}, \frac{b}{c}) = \dfrac{(a^2/c + b^2/c + 1)^2}{a^2/c^2 + b^2/c^2 + 1} = a^2+b^2+c^2$. We are done!

(11) **[6 pts]** Find, *if any*, the local minimum(s), local maximum(s), and saddle point(s) of $f(x,y) = x^2 - xy + y^2 + 9x - 6y + 10$.

$$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \overset{1.5}{\Longrightarrow} \begin{cases} 2x - y + 9 = 0 \\ -x + 2y - 6 = 0 \end{cases} \overset{1.5}{\longrightarrow} \begin{cases} 2x - y = -9 & \longrightarrow x = -4 \\ -x + 2y = 6 & \longrightarrow 3y = 3 \longrightarrow y = 1 \end{cases} \quad 1$$

Only one point : $(-4, 1)$

$f_{xx} = 2$, $f_{xy} = f_{yx} = -1$, $f_{yy} = 2$

$$\Delta = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3 > 0, \; f_{xx} > 0 \quad 1 \longrightarrow \text{minimum} \quad 1$$

(ab. min. indeed!)

$\left(f(-4,1) = -14\right)$

(12) **[6 pts]** Suppose u is a differentiable function of a *single* variable. Show that all tangent planes to the surface $z = f(x,y) = x \cdot u(y/x)$, $(x \neq 0)$, pass through the origin (of $\mathbb{R}^3$).

Let $F(x,y,z) = x \cdot u\left(\frac{y}{x}\right) - z = 0$ 1 , $P(a,b,c)$ such that $F(P) = 0$

$\nabla F(P) = \left(u\left(\frac{b}{a}\right) - \frac{b}{a} \cdot u'\left(\frac{b}{a}\right), \; u'\left(\frac{b}{a}\right), -1 \right)$ 2 ← may be used as a normal to the tangent plane.

The plane : $\nabla F(P) \cdot (x-a, y-b, z-c) = 0$ 2

or $\left(u\left(\frac{b}{a}\right) - \frac{b}{a} u'\left(\frac{b}{a}\right)\right)(x-a) + u'\left(\frac{b}{a}\right)(y-b) - (z-c) = 0$

One easily verifies that $x = y = z = 0$ satisfy the equation of the plane! 1

(13) **[6 pts]** Find the point(s) on the surface (in $\mathbb{R}^3$) with the equation $xy^2z^3 = 4$ that are closest to the origin.

Min $\sqrt{x^2+y^2+z^2}$ subjected to $xy^2z^3=4$, or 1

Min $f = x^2+y^2+z^2$ under $g = xy^2z^3 - 4 = 0$

$$1 \begin{cases} 2x = \lambda y^2 z^3 & (1) \\ 2y = 2\lambda x y z^3 & (2) \\ 2z = 3\lambda x y^2 z^2 & (3) \end{cases}$$

$$\frac{(1)}{(2)} \Rightarrow \frac{x}{y} = \frac{y}{2x} \longrightarrow y^2 = 2x^2 \quad 2$$

$$\frac{(1)}{(3)} \Rightarrow \frac{x}{z} = \frac{z}{3x^2} \longrightarrow z^2 = 3x^2$$

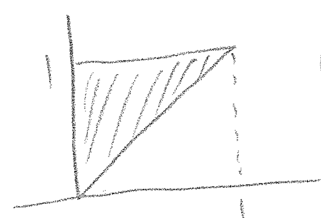
(N.B. $x, y, z, \lambda \neq 0$) Also note that x and z have the same sign. Thus 1 $z = \sqrt{3}\,x$. By replacing, we get

$$x \cdot y^2 \cdot z^3 = 4 \Rightarrow x \cdot \tfrac{1}{2}x^2 \cdot (\sqrt{3}\,x)^3 = 4 \Rightarrow x^6 = \frac{8}{3\sqrt{3}} \Rightarrow x^2 = \frac{2}{\sqrt{3}}$$

$$\Rightarrow x = \pm\left(\frac{2}{\sqrt{3}}\right)^{1/2}, \quad y = \pm\sqrt{2}\left(\frac{2}{\sqrt{3}}\right)^{1/2}, \quad z = \pm\sqrt{3}\left(\frac{\sqrt{2}}{\sqrt{3}}\right)^{1/2}$$

same sign

(14) **[6 pts]** Compute the iterated integral $\int_0^1 \int_x^1 \cos(y^2)\,dy\,dx$.



1

$$= \int_0^1 \int_0^y \cos(y^2)\,dx\,dy \quad 2$$

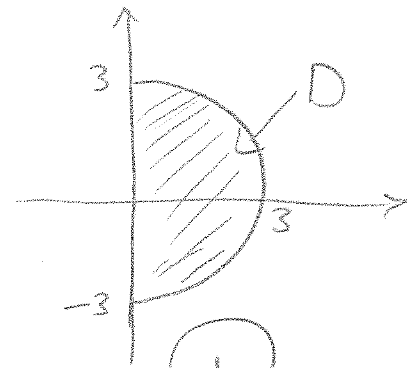
$$= \int_0^1 \Big[\cos(y^2)\,x\Big]_{x=0}^{x=y} dy \quad 1$$

$$= \int_0^1 y\cos(y^2)\,dy = \frac{1}{2}\int_0^1 \cos(u)\,du = \boxed{\frac{1}{2}\sin(1)}$$

1 1

$(u = y^2)$

(15) **[5 pts]** Use polar coordinates to evaluate $\int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (x^3 + xy^2)\,dy\,dx$.



1

$$\iint_D x(x^2+y^2)\,dA = \int_{-\pi/2}^{\pi/2}\int_0^3 r\cos\theta \cdot r^2 \cdot r\,dr\,d\theta$$

2

$$= \int_{-\pi/2}^{\pi/2} \cos(\theta)\,d\theta \cdot \int_0^3 r^4\,dr$$

1

$$= \Big(\sin(\theta)\Big)_{-\pi/2}^{\pi/2} \cdot \left(\frac{1}{5}r^5\right)_0^3 = \frac{2}{5}\cdot 3^5 = \frac{486}{5}$$

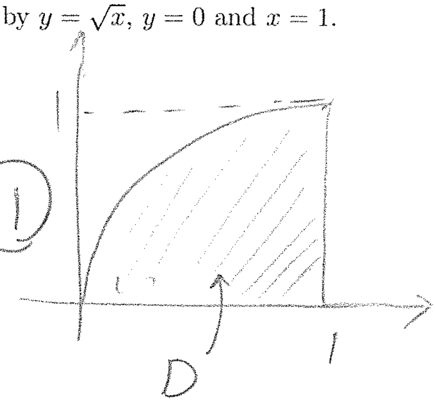
1

(16) **[5 pts]** Compute $\iint_D \frac{y}{1+x^2}\,dA$, where D is the region in $\mathbb{R}^2$ bounded by $y=\sqrt{x}$, $y=0$ and $x=1$.

1

$$\iint_D \frac{y}{1+x^2}\,dA = \int_0^1 \int_0^{\sqrt{x}} \frac{y}{1+x^2}\,dy\,dx$$

1



1

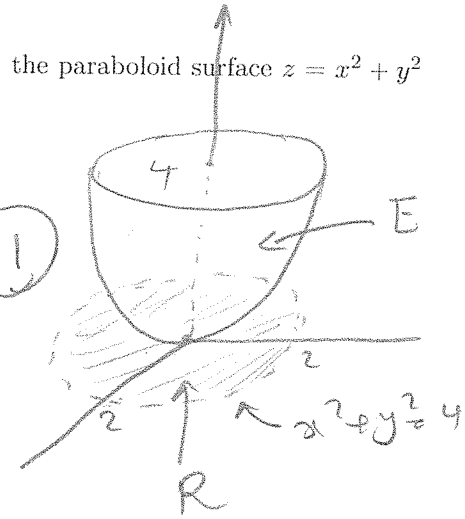
$$= \int_0^1 \frac{1}{1+x^2}\left[\frac{1}{2}y^2\right]_{y=0}^{y=\sqrt{x}}\,dx$$

$$= \frac{1}{2}\int_0^1 \frac{x}{1+x^2}\,dx = \left(\frac{1}{4}\ln(1+x^2)\right)_{x=0}^{x=1} = \boxed{\frac{1}{4}\ln(2)}$$

1

1

(17) **[6 pts]** Compute $\iiint_E z\,dV$, where E is the solid in $\mathbb{R}^3$ enclosed by the paraboloid surface $z = x^2 + y^2$ and the plane $z = 4$.



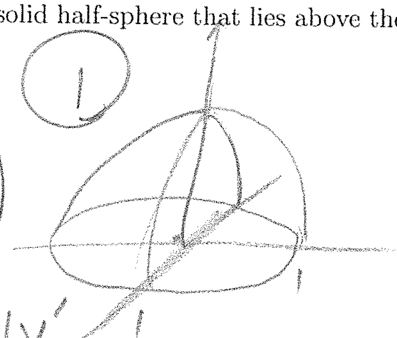
$$\iiint_E z\,dV = \iint_R \left(\int_{x^2+y^2}^{4} z\,dz \right) dA \quad 1$$

$$= \int_0^{2\pi}\int_0^{2} \left[\frac{1}{2}z^2\right]_{z=x^2+y^2}^{4} r\,dr\,d\theta \quad 2$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^{2} \left(16 - (\underbrace{x^2+y^2}_{r^2})^2\right) r\,dr\,d\theta \quad 1$$

$$= \frac{1}{2}\cdot(2\pi)\left(8r^2 - \frac{1}{6}r^6\right)_0^2 = \pi\cdot\left(32 - \frac{32}{3}\right) = \boxed{\frac{64\pi}{3}} \quad 1$$

(18) **[6 pts]** Compute $\iiint_H z^2\sqrt{x^2+y^2+z^2}\,dV$, where H is the solid half-sphere that lies above the xy-plane with center at $(0,0,0)$ and radius 1.



$$H: \begin{cases} 0 \le \varphi \le \pi/2 \\ 0 \le \theta \le 2\pi \\ 0 \le \rho \le 1 \end{cases} \quad 1 \quad \begin{pmatrix} dV = dx\,dy\,dz \\ dV' = d\rho\,d\theta\,d\varphi \end{pmatrix}$$

$$\iiint_H z^2\sqrt{x^2+y^2+z^2}\,dV = \iiint_H \rho^2\cos^2\varphi\cdot\rho\cdot\rho^2\sin\varphi\,dV' \quad 2$$

$$= \int_0^{\pi/2}\left(\int_0^{2\pi}\left(\int_0^1 \rho^5\,d\rho\right)d\theta\right)\cos^2\varphi\sin\varphi\,d\varphi \quad 1$$

$$= (2\pi)\left(\frac{1}{6}\right)\left(-\frac{1}{3}u^3\right)_1^0 = \boxed{\frac{\pi}{9}} \quad 1$$

$u = \cos\varphi$