Dawson College

Mathematics Department

FINAL EXAMINATION

CALCULUS III

201-BZF-05, sections 1 and 2

May 31, 2024 1:30pm – 4:30pm

Student Name: (Solutions)

Student ID #: ____________________

Instructors: A. Hariton, N. Rossokhata

Time: 3 hours

Instructions:

Print your name and student ID number in the space provided above.

All questions are to be answered directly on the examination paper in the space provided. Show your complete work and give explanations.

A Sharp EL-531** calculator is permitted.

This examination consists of 18 questions. Please ensure that you have a complete examination.

This examination must be returned intact.

1. (5 marks)

Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{(2n)!\ (x+3)^n}{(n!)^2}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(2(n+1))!\ (x+3)^{n+1}}{((n+1)!)^2} \cdot \frac{(n!)^2}{(2n)!\ (x+3)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(2n+2)!}{(2n)!} \cdot \frac{(n!)}{(n+1)!} \cdot \frac{(n!)}{(n+1)!} \cdot \frac{(x+3)^{n+1}}{(x+3)^n} \right|$$

$$= \lim_{n\to\infty} \left| (x+3) \frac{(2n+2)(2n+1)}{(n+1)\ (n+1)} \right|$$

$$= |x+3| \lim_{n\to\infty} \frac{4n^2+6n+2}{n^2+2n+1} = 4|x+3|$$

The power series converges if $4|x+3| < 1$

$|x+3| < \frac{1}{4}$

The radius of convergence is $\boxed{R = \frac{1}{4}}$

2. (5 marks)

Find a power series representation for the function

$$f(x) = \frac{x^3}{(4 - x^2)^2}$$

$$\frac{1}{4-x^2} = \frac{\frac{1}{4}}{1-\frac{x^2}{4}} = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)\left(\frac{x^2}{4}\right)^n = \sum_{n=0}^{\infty} \frac{x^{2n}}{4^{n+1}}$$

$$\frac{d}{dx}\left(\frac{1}{4-x^2}\right) = \sum_{n=1}^{\infty} \frac{(2n)x^{2n-1}}{4^{n+1}}$$

$$\frac{2x}{(4-x^2)^2} = \sum_{n=1}^{\infty} \frac{(2n)x^{2n-1}}{4^{n+1}}$$

$$\left(\frac{x^2}{2}\right)\frac{2x}{(4-x^2)^2} = \sum_{n=1}^{\infty} \frac{n x^{2n+1}}{4^{n+1}}$$

$$\frac{x^3}{(4-x^2)^2} = \boxed{\sum_{n=1}^{\infty} \frac{n x^{2n+1}}{4^{n+1}}}$$

3. (5 marks)

Evaluate the definite integral

$$\int_0^1 x^2\, e^{-x^4}\, dx$$

to within an error of 0.01

$$e^{-x^4} = \sum_{n=0}^{\infty} \frac{(-x^4)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{n!}$$

$$x^2 e^{-x^4} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{n!}$$

$$\int x^2 e^{-x^4} dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(4n+3)n!}$$

$$\int_0^1 x^2 e^{-x^4} dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(4n+3)n!} \Bigg|_0^1$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+3)n!}$$

$$= \frac{1}{3} - \frac{1}{7} + \frac{1}{22} - \frac{1}{90} + \frac{1}{456} - \cdots$$

which is an alternating series satisfying the conditions of the Alternating Series Test.

Since $|b_4| = \frac{1}{456} = 0.00219 < 0.01$, we can use:

$$b_0 - b_1 + b_2 - b_3 = \frac{1}{3} - \frac{1}{7} + \frac{1}{22} - \frac{1}{90} = \boxed{0.22}$$

4. (7 marks)

Find the points on the parametric curve $x = t^3 - 3t^2$, $y = 2t^3 - 6t$ where the curve has a horizontal tangent and the points where the curve has a vertical tangent. What happens at the point $(-4, 4)$? Is the curve concave up or concave down at the point $(-2, -4)$?

$$\frac{dx}{dt} = 3t^2 - 6t = 3t(t-2), \qquad \frac{dy}{dt} = 6t^2 - 6 = 6(t^2-1) = 6(t+1)(t-1)$$

The curve has a horizontal tangent where $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$: $6(t+1)(t-1) = 0$, so $\underline{t=-1}$ and $\underline{t=1}$, which correspond to the points $\boxed{(-4,4)}$ and $\boxed{(-2,-4)}$ respectively.

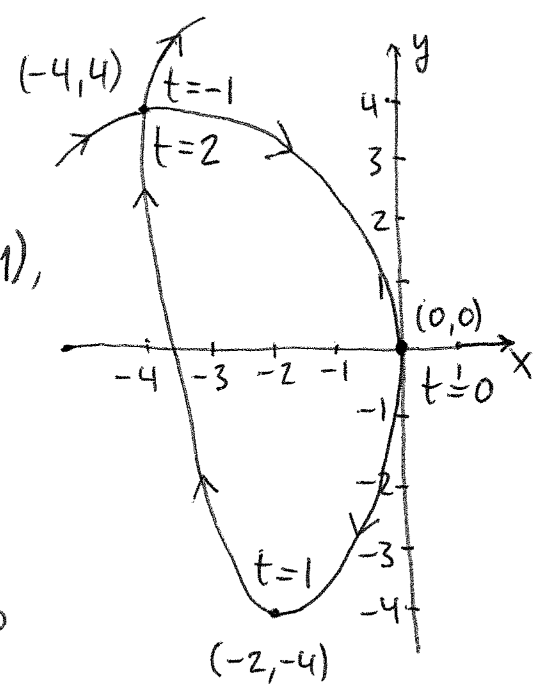
The curve has a vertical tangent where $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$: $3t(t-2) = 0$, so $\underline{t=0}$ and $\underline{t=2}$, which correspond to the points $\boxed{(0,0)}$ and $\boxed{(-4,4)}$ respectively

At point $(-4,4)$, the curve crosses itself, reaching the point at $t=-1$ and again at $t=2$. At point $(-2,-4)$, (where $t=1$),

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{6t^2-6}{3t^2-6t} = \frac{2t^2-2}{t^2-2t}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\left(\frac{dx}{dt}\right)} = \frac{4t(t^2-2t) - (2t-2)(2t^2-2)}{3(t^2-2t)^3}$$

$$= \frac{-4t^2+4t-4}{3(t^2-2t)^3} = \frac{-4+4-4}{3(-1)^3} = \frac{4}{3} > 1,$$ so the curve is $\underline{\text{concave up}}$.



5. (5 marks)

Find the length of the space curve

$$\vec{r}(t) = \langle \tfrac{1}{12}(8t+1)^{3/2}\,,\ 2\cos(t^2)\,,\ 2\sin(t^2)\,\rangle$$

over the interval $0 \le t \le 1$.

$$\vec{r}\,'(t) = \langle \sqrt{8t+1}\,,\ -4t\sin(t^2)\,,\ 4t\cos(t^2) \rangle$$

$$\|\vec{r}\,'(t)\| = \sqrt{(\sqrt{8t+1})^2 + (-4t\sin(t^2))^2 + (4t\cos(t^2))^2}$$

$$= \sqrt{8t+1 + 16t^2\sin^2(t^2) + 16t^2\cos^2(t^2)}$$

$$= \sqrt{8t+1+16t^2} = \sqrt{(4t+1)^2} = 4t+1$$

(since $0 \le t \le 1$)

$$L = \int_0^1 \|\vec{r}\,'(t)\|\,dt = \int_0^1 (4t+1)\,dt = (2t^2+t)\Big|_0^1$$

$$= 2+1 = \boxed{3}$$

6. (6 marks)

Consider the space curve $\vec{r}(t) = \langle \cos(\pi t)\ ,\ 1 + 2\sqrt{t}\,,\ e^{t^2} \rangle$

(a) Find the equation of the tangent line to the curve $\vec{r}(t)$ at the point $(-1, 3, e)$

$$\vec{r}\,'(t) = \langle -\pi \sin(\pi t),\ \frac{1}{\sqrt{t}},\ 2te^{t^2} \rangle$$

At point $(-1,3,e)$, $1+2\sqrt{t}=3$, so $2\sqrt{t}=2$, $\sqrt{t}=1$, $\boxed{t=1}$

At $t=1$, $\vec{r}\,'(1) = \langle 0, 1, 2e \rangle$

The equation of the tangent line at point $(-1,3,e)$ is:

$\langle x, y, z \rangle = \langle -1, 3, e \rangle + t \langle 0, 1, 2e \rangle$, that is:

$$\boxed{x = -1,\quad y = 3+t,\quad z = e(1+2t)}$$

(b) Find the equation of the normal plane to the curve $\vec{r}(t)$ at the point $(-1, 3, e)$

The normal plane is the plane which is perpendicular to the tangent vector $\vec{r}\,'(1) = \langle 0, 1, 2e \rangle$.

Its equation is: $a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$

$$(0)(x+1) + (1)(y-3) + 2e(z-e) = 0$$

$$\boxed{y + 2ez - 3 - 2e^2 = 0}$$

7. (8 marks)

A particle has an acceleration of $\vec{a}(t) = \langle -2\ ,\ 6t\ ,\ 0\ \rangle$, an initial position of $\vec{r}(0) = \langle 2\ ,-3\ ,\ 0\ \rangle$ and an initial velocity of $\vec{v}(0) = \langle 0\ ,\ 0\ ,\ 1\ \rangle$

(a) Find the position function $\vec{r}(t)$ of the particle

$\vec{v}(t) = \int \vec{a}(t)\,dt = \int \langle -2, 6t, 0\rangle dt = \langle -2t + C_1, 3t^2 + C_2, C_3\rangle$

$\vec{v}(0) = \langle C_1, C_2, C_3\rangle = \langle 0,0,1\rangle$, so $C_1 = 0$, $C_2 = 0$, $C_3 = 1$

$\boxed{\vec{v}(t) = \langle -2t, 3t^2, 1\rangle}$ velocity function

$\vec{r}(t) = \int \vec{v}(t)\,dt = \int \langle -2t, 3t^2, 1\rangle dt = \langle -t^2 + K_1, t^3 + K_2, t + K_3\rangle$

$\vec{r}(0) = \langle K_1, K_2, K_3\rangle = \langle 2, -3, 0\rangle$, so $K_1 = 2$, $K_2 = -3$, $K_3 = 0$

$\boxed{\vec{r}(t) = \langle -t^2 + 2\ ,\ t^3 - 3\ ,\ t\ \rangle}$ position function

(b) Find the curvature of the space curve defined by the position function of the particle at the point $(1, -2, 1)$

At point $(1,-2,1)$, $\underline{t=1}$, so $\vec{r}\,'(1) = \vec{v}(1) = \langle -2, 3, 1\rangle$

$\vec{r}\,''(1) = \vec{a}(1) = \langle -2, 6, 0\rangle$

$\|\vec{r}\,'(1)\| = \sqrt{(-2)^2 + (3)^2 + (1)^2} = \sqrt{14}$

$\vec{r}\,'(1) \times \vec{r}\,''(1) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 3 & 1 \\ -2 & 6 & 0 \end{vmatrix} = \langle -6, -2, -6\rangle$

$\|\vec{r}\,'(1) \times \vec{r}\,''(1)\| = \sqrt{(-6)^2 + (-2)^2 + (-6)^2} = \sqrt{76} = 2\sqrt{19}$

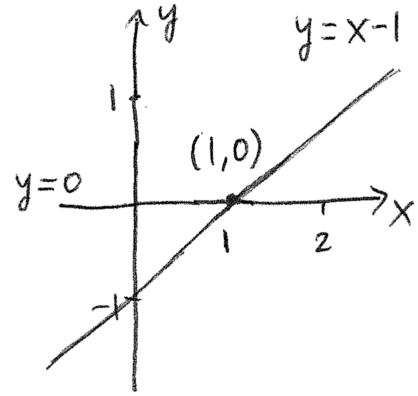
Curvature : $K(1) = \dfrac{\|\vec{r}\,'(1) \times \vec{r}\,''(1)\|}{\|\vec{r}\,'(1)\|^3} = \dfrac{2\sqrt{19}}{(\sqrt{14})^3} = \boxed{\dfrac{\sqrt{19}}{7\sqrt{14}}}$

8. (5 marks)

Find the limit

$$\lim_{(x,y)\to(1,0)} \frac{xy-y}{(x-1)^2+2y^2}$$

or show that the limit does not exist. Justify your answer.



The x-axis ($y=0$) passes through $(1,0)$.

Along $y=0$, as $x\to 1$,

$$\frac{xy-y}{(x-1)^2+2y^2} = \frac{0}{(x-1)^2} = \boxed{0} \to \boxed{0}$$

The line $y=x-1$ passes through $(1,0)$.

Along $y=x-1$, as $x\to 1$.

$$\frac{xy-y}{(x-1)^2+2y^2} = \frac{(x-1)y}{(x-1)^2+2y^2} = \frac{(x-1)^2}{(x-1)^2+2(x-1)^2}$$

$$= \frac{(x-1)^2}{3(x-1)^2} = \boxed{\frac{1}{3}} \to \boxed{\frac{1}{3}}$$

Since the function approaches two different values along two different paths approaching $(1,0)$, the limit does not exist.

9. (6 marks)

Find the equation of the tangent plane to the surface $z = y\ln(xy-5) - 3x$ at the point $(3, 2, -9)$. Also find the linearization $L(x, y)$ of the function $f(x, y) = y\ln(xy-5) - 3x$ at the point $(3, 2)$. Use this linearization to estimate the value of f at the point $(3.1, 1.9)$ by calculating $L(3.1, 1.9)$

If $z = f(x,y) = y\ln(xy-5) - 3x$, then:

$$f_x = \frac{y^2}{xy-5} - 3, \quad \text{so at } (3,2), \quad f_x = \frac{(2)^2}{(3)(2)-5} - 3 = 1$$

$$f_y = \ln(xy-5) + \frac{xy}{xy-5}, \quad \text{so at } (3,2),$$

$$f_y = \ln((3)(2)-5) + \frac{(3)(2)}{(3)(2)-5} = 6$$

The equation of the tangent plane at $(3,2,-9)$ is:

$$z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$$z + 9 = (1)(x-3) + (6)(y-2)$$

$$z + 9 = x - 3 + 6y - 12$$

$$\boxed{x + 6y - z - 24 = 0}$$

The linearization at point $(3,2)$ is:

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$$= -9 + (1)(x-3) + (6)(y-2)$$

$$= \boxed{x + 6y - 24}$$

At point $(3.1, 1.9)$, $L(3.1, 1.9) = 3.1 + 6(1.9) - 24 = \boxed{-9.5}$

10. (6 marks)

If $z = 2xe^y + y\sin x$, where $x = r^2 s + 5t$ and $y = \ln t + r$,

then find $\dfrac{\partial z}{\partial r}$, $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$

(the answers can be given in terms of x, y, r, s and t)

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \boxed{(2e^y + y\cos x)(2rs) + (2xe^y + \sin x)}$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = \boxed{(2e^y + y\cos x)(r^2)}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = \boxed{(2e^y + y\cos x)(5) + (2xe^y + \sin x)\left(\frac{1}{t}\right)}$$

11. (7 marks)

Let $f(x,y,z) = xy^2 + z\ln(x+yz)$

(a) Find the directional derivative of f at the point $(3,-2,1)$ in the direction of the vector $\vec{v} = \langle 2,3,-6\rangle$

$$\vec{\nabla} f(x,y,z) = \left\langle y^2 + \frac{z}{x+yz},\ 2xy + \frac{z^2}{x+yz},\ \ln(x+yz) + \frac{yz}{x+yz} \right\rangle$$

$$\vec{\nabla} f(3,-2,1) = \left\langle (-2)^2 + \frac{1}{3+(-2)(1)},\ 2(3)(-2) + \frac{(1)^2}{3+(-2)(1)},\ \ln(3+(-2)(1)) + \frac{(-2)(1)}{3+(-2)(1)} \right\rangle$$

$$= \langle 4+1, -12+1, \ln(1)-2\rangle = \langle 5,-11,-2\rangle$$

$$\vec{u} = \frac{1}{\sqrt{2^2+3^2+(-6)^2}}\langle 2,3,-6\rangle = \frac{1}{\sqrt{49}}\langle 2,3,-6\rangle = \left\langle \frac{2}{7}, \frac{3}{7}, -\frac{6}{7}\right\rangle$$

so $D_{\vec{u}} f(3,-2,1) = \vec{\nabla} f(3,-2,1)\cdot\vec{u} = \langle 5,-11,-2\rangle \cdot \left\langle \frac{2}{7}, \frac{3}{7}, -\frac{6}{7}\right\rangle$

$$= \boxed{-\frac{11}{7}}$$

(b) Find the maximum rate of change of the function f at the point $(3,-2,1)$

$$\|\vec{\nabla} f(3,-2,1)\| = \sqrt{5^2+(-11)^2+(-2)^2} = \sqrt{150} = \boxed{5\sqrt{6}}$$

12. (5 marks)

Find the local maxima, local minima and saddle points of the function

$f(x, y) = x^4 + y^2 + 2xy$

$f_x = 4x^3 + 2y$, $f_y = 2y + 2x$

If $f_y = 0$, then $2y + 2x = 0$, so $\underline{y = -x}$

If $f_x = 0$, then $4x^3 + 2(-x) = 0$

$2x(2x^2 - 1) = 0$

$\underline{x = 0}$ or $2x^2 - 1 = 0$, $2x^2 = 1$, $x^2 = \frac{1}{2}$,

$\underline{x = \pm\frac{1}{\sqrt{2}}}$

critical points: $(0,0)$, $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$, $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

$f_{xx} = 12x^2$, $f_{yy} = 2$, $f_{xy} = 2$, so

$D = f_{xx} f_{yy} - f_{xy}^2 = \underline{24x^2 - 4}$

At $(0,0)$: $D = -4 < 0$, so $\boxed{(0,0) \text{ is a saddle point}}$

At $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$: $D = 24(\frac{1}{\sqrt{2}})^2 - 4 = 24(\frac{1}{2}) - 4 = 8 > 0$

$f_{xx} = 12(\frac{1}{\sqrt{2}})^2 = 12(\frac{1}{2}) = 6 > 0$

so we have a $\boxed{\text{local minimum at } (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) \text{ with value } -\frac{1}{4}}$

At $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$: $D = 24(-\frac{1}{\sqrt{2}})^2 - 4 = 12 - 4 = 8 > 0$

$f_{xx} = 12(-\frac{1}{\sqrt{2}})^2 = 6 > 0$

so we have a $\boxed{\text{local minimum at } (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \text{ with value } -\frac{1}{4}}$

13. (5 marks)

Find the maximum and minimum values of the function $f(x, y) = e^{xy}$ subject to the constraint $x^2 + 4y^2 = 1$

$$ye^{xy} = 2x\lambda \quad ①$$

$$xe^{xy} = 8y\lambda \quad ②$$

$$x^2 + 4y^2 = 1 \quad ③$$

If $\lambda = 0$, then ① and ② give $x=0$ and $y=0$, which contradicts ③, so $\boxed{\lambda \neq 0}$.

from ①: $xye^{xy} = 2x^2\lambda$
from ②: $xye^{xy} = 8y^2\lambda$
$\Big\}$ so $2x^2\lambda = 8y^2\lambda$

$\boxed{x^2 = 4y^2}$ since $\lambda \neq 0$.

into ③: $4y^2 + 4y^2 = 1$, so $8y^2 = 1$, $y^2 = \frac{1}{8}$, $\boxed{y = \pm\frac{1}{2\sqrt{2}}}$

Also, $x^2 = 4y^2 = 4\left(\frac{1}{8}\right) = \frac{1}{2}$, so $\boxed{x = \pm\frac{1}{\sqrt{2}}}$

Four points $\left(\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$, $\left(\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}}\right)$, $\left(-\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$, $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}}\right)$

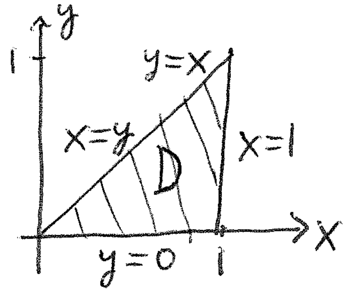
$f\left(\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right) = f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}}\right) = \boxed{e^{\frac{1}{4}}}$ (maximum)

$f\left(\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}}\right) = f\left(-\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right) = \boxed{e^{-\frac{1}{4}}}$ (minimum)

14. **(5 marks)**

Evaluate the iterated integral

$$\int_0^1 \int_y^1 y^3\, e^{x^5}\; dx\; dy$$



$$\int_0^1 \int_y^1 y^3 e^{x^5}\, dx\, dy = \iint_D y^3 e^{x^5}\, dA,$$

where $D = \{(x,y) \mid 0 \le y \le 1, \quad y \le x \le 1\}$

which is also $D = \{(x,y) \mid 0 \le x \le 1, \quad 0 \le y \le x\}$

so $\int_0^1 \int_y^1 y^3 e^{x^5}\, dx\, dy = \int_0^1 \int_0^x y^3 e^{x^5}\, dy\, dx$

$= \int_0^1 \frac{1}{4} y^4 e^{x^5} \Big|_{y=0}^{y=x} dx = \int_0^1 \frac{1}{4} x^4 e^{x^5} dx$

$= \frac{1}{20} e^{x^5} \Big|_0^1 = \frac{1}{20} e^{(1)} - \frac{1}{20} e^{(0)} = \boxed{\frac{1}{20}(e-1)}$

15. **(5 marks)**

Evaluate the double integral

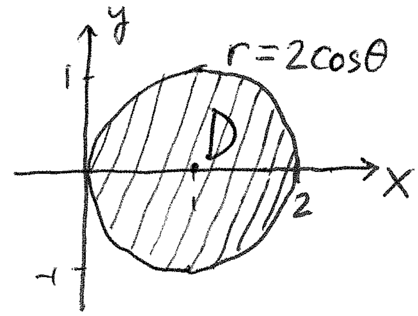
$$\iint_D x\, dA$$

where D is the region enclosed by the curve $x^2 + y^2 = 2x$

$x^2+y^2=2x$ is the circle

$x^2-2x+1+y^2=1$

$(x-1)^2+y^2=1$ (of radius 1, centered at $(1,0)$)



In polar coordinates, this circle is $r^2 = 2r\cos\theta$

or $\boxed{r = 2\cos\theta}$

$D = \{(r,\theta) \mid -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},\ 0 \le r \le 2\cos\theta\}$

$$\iint_D x\, dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} (r\cos\theta)\, r\, dr\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} r^2\cos\theta\, dr\, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{3} r^3 \cos\theta \Big|_{r=0}^{r=2\cos\theta} d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{8}{3}\cos^4\theta\, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{8}{3}\left(\frac{1}{2}(1+\cos(2\theta))\right)^2 d\theta$$

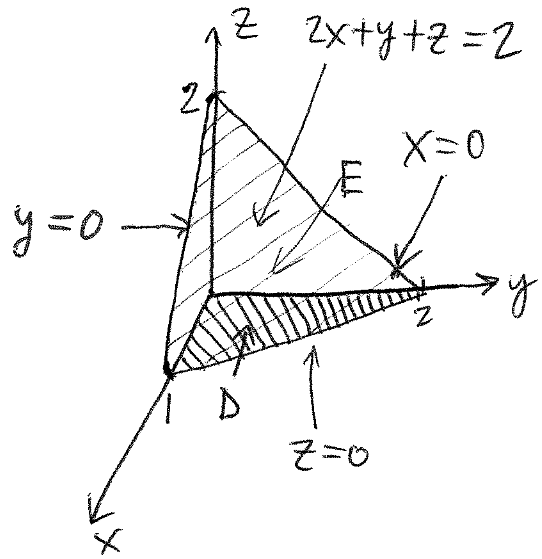
$$= \frac{2}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(1 + 2\cos(2\theta) + \frac{1}{2} + \frac{1}{2}\cos(4\theta)\right) d\theta$$

$$= \frac{2}{3}\left(\frac{3}{2}\theta + \sin(2\theta) + \frac{1}{8}\sin(4\theta)\right)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \boxed{\pi}$$

16. (5 marks)

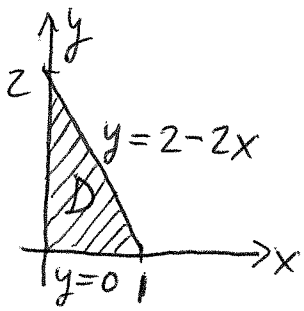
Find the volume of the tetrahedron bounded by the four planes $x = 0$, $y = 0$, $z = 0$ and $2x + y + z = 2$

The tetrahedron occupies the solid region E given by



$$E = \{ (x,y,z) \mid 0 \le x \le 1,\ 0 \le y \le 2-2x,\ 0 \le z \le 2-2x-y \}$$

so the volume of the tetrahedron is:



$$V = \iiint_E (1)\, dV$$

$$= \int_0^1 \int_0^{2-2x} \int_0^{2-2x-y} dz\, dy\, dx$$

$$= \int_0^1 \int_0^{2-2z} z \Big|_{z=0}^{z=2-2x-y} dy\,dx = \int_0^1 \int_0^{2-2x} (2-2x-y)\, dy\, dx$$

$$= \int_0^1 \left(2y - 2xy - \tfrac{1}{2}y^2\right) \Big|_{y=0}^{y=2-2x} dx$$

$$= \int_0^1 \left(2(2-2x) - 2x(2-2x) - \tfrac{1}{2}(4 - 8x + 4x^2)\right) dx$$

$$= \int_0^1 (4 - 4x - 4x + 4x^2 - 2 + 4x - 2x^2)\, dx$$

$$= \int_0^1 (2x^2 - 4x + 2)\, dx = \left(\tfrac{2}{3}x^3 - 2x^2 + 2x\right)\Big|_0^1 = \boxed{\tfrac{2}{3}}$$

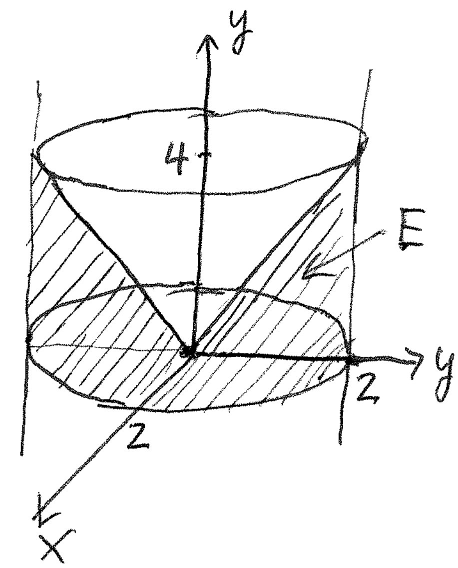
17. (5 marks)

Use cylindrical coordinates to evaluate the triple integral

$$\iiint_E y^2 \, dV$$

where E is the solid that lies within the cylinder $x^2 + y^2 = 4$, above the plane $z = 0$ and below the half-cone $z = \sqrt{4x^2 + 4y^2}$

In cylindrical coordinates, the cylinder $x^2+y^2=4$ has equation $\boxed{r=2}$ and the half-cone $z=\sqrt{4x^2+4y^2}$ has equation $z=\sqrt{4r^2}$ or $\boxed{z=2r}$



$$E = \{(r,\theta,z) \mid 0 \le \theta \le 2\pi,\ 0 \le r \le 2,\ 0 \le z \le 2r\}$$

$$\iiint_E y^2\,dV = \int_0^{2\pi}\int_0^2\int_0^{2r} (r\sin\theta)^2\, r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2\int_0^{2r} r^3\sin^2\theta\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 r^3\sin^2\theta\, z\Big|_{z=0}^{z=2r} dr\,d\theta = \int_0^{2\pi}\int_0^2 2r^4\sin^2\theta\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{2}{5} r^5 \sin^2\theta \Big|_{r=0}^{r=2} d\theta = \int_0^{2\pi} \frac{64}{5}\,\frac{1}{2}(1-\cos(2\theta))\,d\theta$$

$$= \frac{32}{5}\left(\theta - \frac{1}{2}\sin(2\theta)\right)\Big|_0^{2\pi} = \frac{32}{5}(2\pi) = \boxed{\frac{64\pi}{5}}$$

18. (5 marks)

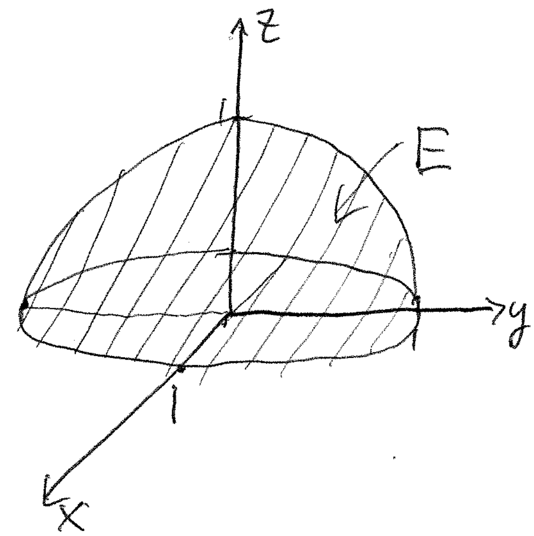
Use spherical coordinates to evaluate the triple integral

$$\iiint_E z\, e^{(x^2+y^2+z^2)^2}\, dV$$

where $E = \{(x,y,z) \mid x^2+y^2+z^2 \le 1,\ \ z \ge 0\}$.

In spherical coordinates,

$z = \rho\cos\phi$ and $x^2+y^2+z^2 = \rho^2$.



$$E = \{(\rho,\theta,\phi) \mid 0 \le \theta \le 2\pi,\ 0 \le \phi \le \tfrac{\pi}{2},\ 0 \le \rho \le 1\}$$

$$\iiint_E z\, e^{(x^2+y^2+z^2)^2} dV = \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^1 \rho\cos\phi\, e^{\rho^4}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^1 \rho^3 e^{\rho^4}\cos\phi\sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \frac{1}{4}e^{\rho^4}\Big|_{\rho=0}^{\rho=1}\cos\phi\sin\phi\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \frac{1}{4}(e-1)\cos\phi\sin\phi\, d\phi\, d\theta$$

$$= \int_0^{2\pi} \frac{1}{8}(e-1)\sin^2\phi\Big|_{\phi=0}^{\phi=\frac{\pi}{2}} d\theta$$

$$= \int_0^{2\pi}\frac{1}{8}(e-1)\, d\theta = \frac{1}{8}(e-1)(2\pi) = \boxed{\frac{\pi}{4}(e-1)}$$