DAWSON COLLEGE
Mathematics Department
Final Examination
Calculus III
201-BZF-05 Sections 01, 02
May 18^{th}, 2018

Student Name ______________________________

Student I.D. # ______________________________

Teachers: **R. Fournier, A. Gambioli**

TIME: 2:00 pm – 5:00 pm

Instructions:

- Print your name and student I.D. number in the space provided above.All questions are to be answered directly on the examination paper in the provided space.
- Translation and regular dictionaries are permitted.
- The only calculators permitted during the exam are the Sharp EL 531 X, XG and XT.
- This examination consists of 20 problems.
- There are 12 pages including the cover page.
- **This exam booklet must be returned intact.**

50% Class Marks = ________
+
50% Final Exam = ________
FINAL GRADE = ________

Question #	Marks
1 (5 marks)	
2(5 marks)	
3 (5 marks)	
4 (5 marks)	
5 (5 marks)	
6 (5 marks)	
7 (5 marks)	
8 (5 marks)	
9 (5 marks)	
10 (5 marks)	
12 (5 marks)	
13 (5 marks)	
14 (5 marks)	
15 (5 marks)	
16 (5 marks)	
17 (5 marks)	
18 (5 marks)	
19 (5 marks)	
20 (5 marks)	
Total / 100	

Problem 1.(5 marks) Find a Power Series representation and the radius of convergence for

$$\frac{x^2}{(x^2+2)^2}\,.$$

$$\frac{1}{x+2} = \frac{1}{2}\,\frac{1}{1+x/2} = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{2^n} \qquad R=2$$

$$\frac{-1}{(x+2)^2} = \frac{d}{dx}\,\frac{1}{x+2} = \frac{1}{2}\sum_{n=1}^{\infty}(-1)^n\frac{n\,x^{n-1}}{2^n} \qquad R=2$$

$$\frac{1}{(x^2+2)^2} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n\,x^{2n-2}}{2^n} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{n\,x^{2n-2}}{2^{n+1}} \qquad R=\sqrt{2}$$

$$\frac{x^2}{(x^2+2)^2} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}n}{2^{n+1}}\,x^{2n} \qquad R=\sqrt{2}$$

Problem 2.(5 marks) Use series to approximate the value of the integral with an error less than 0.001.

$$\int_0^1 x^5 e^{-x^{10}}\,dx\,.$$

$$x^5 e^{-x^{10}} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\,x^{10n+5} \qquad R=\infty$$

$$\int_0^1 x^5 e^{-x^{10}}\,dx = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\,\frac{1}{10n+6}$$

$$\left|\int_0^1 x^5 e^{-x^{10}}\,dx - \sum_{n=0}^{N}\frac{(-1)^n}{n!}\,\frac{1}{10n+6}\right| \le \frac{1}{(N+1)!\,(10N+16)} \le \frac{1}{1000}$$

if $(N+1)!\,(10N+16) \geq 1000$

find N with the calculator and then this is the approximation

Problem 3.(5 marks) Evaluate the limit

$$\lim_{x\to 0^+} \frac{x^5 - \sin x^5}{x^{10}}.$$

This equals $\lim\limits_{x\to 0^+} \dfrac{x - \sin x}{x^2} = \lim\limits_{x\to 0^+} \dfrac{x - \sum\limits_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}}{x^2}$ $R = \infty$

$$= \lim_{x\to 0^+} - \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n-1} = 0$$

because of continuity since all exponents are strictly positive

Problem 4.(5 marks) True or False? Answer with a short justification: Any Power Series $\sum_0^{+\infty} a_n x^n$ convergent over $[-1, 1)$ is also absolutely convergent on the same interval.

False $\sum\limits_{n=1}^{\infty} \frac{1}{n} x^n$ converges for $x \in [-1, 1)$

but is not absolutely convergent there

because $\sum\limits_{n=1}^{\infty} \left| \frac{1}{n} (-1)^n \right| = \sum\limits_{n=1}^{\infty} \frac{1}{n} = \infty$

Problem 5.(5 marks) If a curve has parametric equations

$$x = t + \ln(1-t) \quad ; \quad y = t - \ln(1-t) \quad , \quad 0 \le t < 1$$

find the equation of the tangent line at $P = (0,0)$.

The directon vector of the tangent line at $(0,0)$ (i.e. $t=0$) is $(x'(0), y'(0)) = \left(1 - \frac{1}{1-t}, 1 + \frac{1}{1-t}\right)_{t=0}$

$= (0, 1)$

so that the tangent line is a vertical line with equation $x=0$ (in $\mathbb{R}^2$)

Problem 6.(5 marks) Compute at any point the curvature $k = \frac{|r' \times r''|}{|r'|^3}$ for the helix in $\mathbb{R}^3$ with equation $r(t) = (\cos(2t), \sin(2t), 2t)$, $t \ge 0$.

$r'(t) = (-2\sin(2t), 2\cos(2t), 2)$

$r''(t) = (-4\cos(2t), -4\sin(2t), 0)$

$$k = \frac{|r' \times r''(t)|}{|r'(t)|^3} = \frac{|(8\sin(2t), -8\cos(2t), 8)|}{|(-2\sin 2t, 2\cos(2t), 2)|^3}$$

$$= \frac{\sqrt{128}}{(\sqrt{8})^3} = \frac{8\sqrt{2}}{8\sqrt{8}} = \frac{\sqrt{2}}{\sqrt{8}} = \frac{\sqrt{2}}{2\sqrt{2}} = \frac{1}{2}$$

the curvature is constant

Problem 7.(5 marks) Find the arc-length parametrization for the curve with equation

$$r(t) = \left(\frac{1-4t^2}{1+4t^2}, \frac{4t}{1+4t^2}\right)$$

if $t \geq 0$. What can you say about the curve?

$$r'(x) = \left(\frac{-16x}{(1+4x^2)^2}, \frac{4(1-4x^2)}{(1+4x^2)^2}\right) = \frac{4}{(1+4x^2)^2}\left(-4x, 1-4x^2\right)$$

$$|r'(x)| = \frac{4}{(1+4x^2)^2}\sqrt{16x^2+(1-4x^2)^2} = \frac{4(1+4x^2)}{(1+4x^2)^2} = \frac{4}{1+4x^2}$$

$$s = \int_0^t |r'(x)|\,dx = 4\int_0^t \frac{1}{1+(2x)^2}\,dx = 2\arctan(2t)$$

$$t \geq 0$$

$$2t = \tan\left(\frac{s}{2}\right), \quad 0 \leq s \leq \pi$$

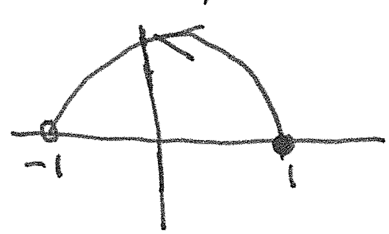
$$r(s) = \left(\frac{1-(2t)^2}{1+(2t)^2}, \frac{2(2t)}{1+(2t)^2}\right)$$

$$= \left(\frac{1-\tan^2\left(\frac{s}{2}\right)}{1+\tan^2\left(\frac{s}{2}\right)}, \frac{2\tan\left(\frac{s}{2}\right)}{1+\tan^2\left(\frac{s}{2}\right)}\right)$$

$$= \left(\cos^2\left(\frac{s}{2}\right)-\sin^2\left(\frac{s}{2}\right), 2\sin\left(\frac{s}{2}\right)\cos\left(\frac{s}{2}\right)\right)$$

$$= \left(\cos(s), \sin(s)\right), \quad 0 \leq s \leq \pi$$

It is a half-circle



Problem 8.(5 marks) Prove that the circle in $\mathbb{R}^3$ parametrized by

$$x(t) = (\sin(2t), \cos(2t), 0)$$

has constant curvature. Is it the only curve in $\mathbb{R}^3$ with constant curvature?

$$x'(t) = (2\cos(2t), -2\sin(2t), 0)$$

$$x''(t) = (-4\sin(2t), -4\cos(2t), 0)$$

$$K = \frac{|x' \times x''(t)|}{|x'(t)|^3} = \frac{|(0, 0, -8)|}{(\sqrt{4})^3} = \frac{8}{8} = 1$$

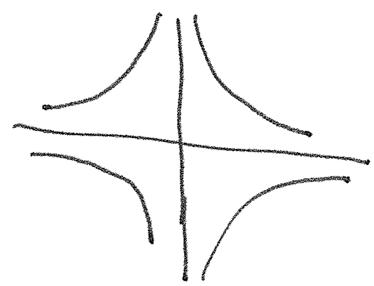
i.e. the curvature of a circle of radius 1 is 1

It is not the only one : see the curve (helix!) of problem 6

Problem 9.(5 marks) Study the continuity of the function:

$$f(x,y) = \begin{cases} \frac{-\ln(1-x^2y^2)}{x^2y^2} & xy \neq 0 \\ 1 & xy = 0 \end{cases}$$

The domain is the region inside four hyperbolas $|xy| < 1$



It is rather clear that f is continuous at any (x_0, y_0) with $x_0 y_0 \neq 0$ (quotient of two continuous functions. Moreover at a point P where $x_0 y_0 = 0$, we have

$$\lim_{(x,y)\to P} \frac{-\ln(1-x^2y^2)}{x^2y^2} = \lim_{t\to 0^+} \frac{-\ln(1-t)}{t} = 1 = f(P)$$

Problem 10.(5 marks) Find and classify the critical points of

$$f(x,y) = 2 - x^4 + 2x^2 - y^2 .$$

$$\frac{\partial f}{\partial x} = -4x^3 + 4x = 4x(1-x^2) = 0 \quad \text{if} \quad x = 0, \pm 1$$

$$\frac{\partial f}{\partial y} = -2y = 0 \quad \text{if} \quad y = 0$$

There are 3 critical points $(0,0)$, $(1,0)$, $(-1,0)$

At $(0,0)$ we have $\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 - \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} = 0 - (4 - 3(0)^2)(-2)$
$= 8 > 0$

and $(0,0)$ is a saddle point.

At $(\pm 1, 0)$ we have $\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 - \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} = 0 - (4 - 12(\pm 1)^2)(-2)$
$= 8(1-3) < 0$

with $\frac{\partial^2 f}{\partial x^2}(\pm 1, 0) = -12(1)^2 + 4 < 0$ so $(\pm 1, 0)$ is a point of local max

Problem 11.(5 marks) Find the absolute maximum and minimum of the function

$$f(x,y) = \sqrt{x^2 + y^2}$$

on the square $[-1,1] \times [-1,1]$.

This can be done by finding critical points inside the square (i.e. $(0,0)$) and studying the function on the boundary of the square ...

But clearly, geometry yields that

$$0 \leq f(x,y) \leq f(1,1) = f(-1,-1) = f(-1,1) = f(1,1) = \sqrt{2}$$

here equality holds iff. $(x,y) = (0,0)$

here equality holds only if $(x,y) = (\pm 1, \pm 1)$

Problem 12.(5 marks) Maximize the function $f(x, y, z) = \sqrt{xyz}$ under the constraints $x + y + z = 1$, $x \geq 0$, $y \geq 0$ and $z \geq 0$.

By Lagrange, at a solution point, we have

$$\nabla f + L \nabla g = (0,0,0) \quad \text{for some}$$

L with $f(x,y,z) = \sqrt{xyz}$ and $g(x,y,z) = x+y+z-1=0$

i.e. $\dfrac{(yz)^{1/2}}{2x^{1/2}} = \dfrac{(xz)^{1/2}}{2y^{1/2}} = \dfrac{(xy)^{1/2}}{2z^{1/2}}$ and clearly $x=y=z=1/3$

so that the maximum is $\sqrt{\left(\frac{1}{3}\right)^3} = \sqrt{\frac{1}{27}}$

Problem 13.(5 marks) If $f(x, y, z) = x + y^2 + z^2$ with $x = uvw$, $y = u^2v^2w^2$ and $z = u^3v^3w^3$, compute $\frac{\partial f}{\partial w}$ at $(u, v, w) = (1, 2, 3)$

$$\frac{\partial f}{\partial w} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial w} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial w} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial w}$$

$$= 1\,(uv) + 2y\,(2u^2v^2w) + 2z\,(3u^3v^3w^2)$$

$$= uv + 2(u^2v^2w^2)(2u^2v^2w) + 2u^3v^3w^3(3u^3v^3w^2)$$

Now set $u=1$, $v=2$, $w=3$

Problem 14.(5 marks) Find the directional derivative of

$$f(x,y,z) = \frac{x^2}{1+x^2} + \frac{y^2}{1+y^2} + \frac{z^2}{1+z^2}$$

at the point $(1,1,1)$ in the direction of maximal increase.

the direction of maximal increase is the direction of the gradient and there

$$D_{\frac{\nabla f(1,1,1)}{|\nabla f(1,1,1)|}} f(1,1,1) = \nabla f(1,1,1) \cdot \frac{\nabla f(1,1,1)}{|\nabla f(1,1,1)|}$$

$$= |\nabla f(1,1,1)|$$

$$= \left| \left(\frac{2x}{(1+x^2)^2}, \frac{2y}{(1+y^2)^2}, \frac{2z}{(1+z^2)^2} \right) \right|_{x=y=1}$$

$$= |(1/2, 1/2, 1/2)| = \sqrt{3/4} = \frac{\sqrt{3}}{2}$$

Problem 15.(5 marks) Compute $\frac{\partial^3 f}{\partial x \partial y^2}$ and $\frac{\partial^3 f}{\partial y \partial x \partial y}$ for

$$f(x,y,z) = x + xy^2 + xyz^3$$

at the point $(1,2,3)$.

Because of the theorem of Schwarz, these are equal

$$\frac{\partial f}{\partial y} = 2xy + xz^3$$

$$\frac{\partial^2 f}{\partial y^2} = 2x$$

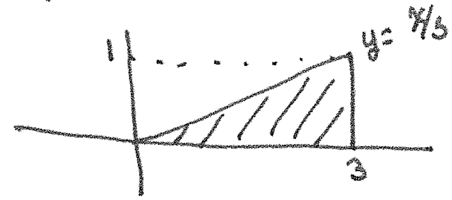
$$\frac{\partial^3 f}{\partial x \partial y^2} = 2$$

Problem 16.(5 marks) Evaluate the integral by reversing the order of integration:

$$\int_0^1 \left(\int_{3y}^3 e^{x^2}\, dx \right) dy\ .$$

By Fubini the iterated integral is

$\iint_T e^{x^2}\, dA$ where



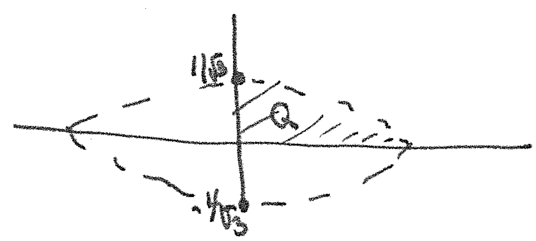
This is also, by Fubini,

$$\int_0^3 \left(\int_0^{x/3} e^{x^2} dy \right) dx = \frac{1}{3} \int_0^3 x e^{x^2} dx = \frac{e^{x^2}}{6} \Big]_0^3 = \frac{e^9 - 1}{6}$$

Problem 17.(5 marks) Using a double integral, compute the area of the ellipse with equation $x^2 + 3y^2 = 1$.

$$x^2 + y^2/_{1/3} = 1$$

$$\left(\frac{x}{1}\right)^2 + \frac{y^2}{(1/\sqrt{3})^2} = 1$$



By sym. the area is

$$4 \iint_Q dA = 4 \int_0^1 \frac{\sqrt{1-x^2}}{\sqrt{3}}\, dx = \frac{4}{\sqrt{3}} \times \text{area of a quarter circle of radius 1}$$

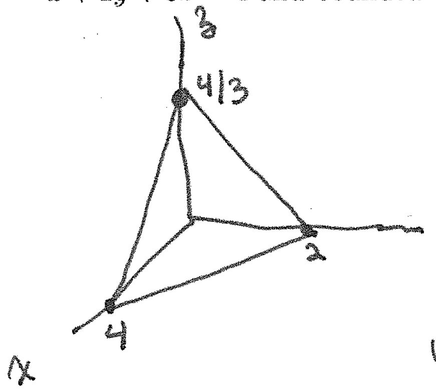
$$= \frac{4\pi/4}{\sqrt{3}} = \pi/\sqrt{3}$$

Problem 18.(5 marks) Compute the integral $\iint_D \sqrt{x^2+y^2}\, dA$ where D is the disk centered at the origin with radius 1.

By using polar coordinates, the integral equals

$$\iint_{[0,1]\times[0,2\pi]} r\, r\, dA' = \int_0^1 \left(\int_0^{2\pi} r^2 d\theta \right) dr = \frac{2\pi}{3}$$

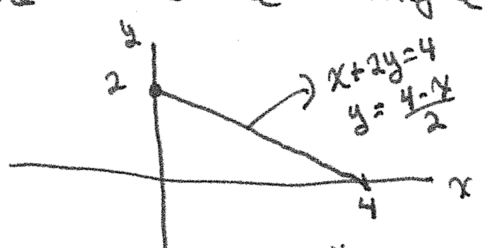
Problem 19.(5 marks) Compute the volume of the solid lying under the plane $x+2y+3z=4$ and bounded by the 3 coordinate planes in the first octant.



Clearly the integral (volume) is

$$\iint_T \frac{4-x-2y}{3}\, dA$$

where T is the triangle



and Fubini yields

$$\int_0^4 \left(\int_0^{\frac{4-x}{2}} \left(\frac{4-x}{3} - \frac{2}{3} y \right) dy \right) dx$$

$$= \int_0^4 \left[\left(\frac{4-x}{3}\right)\left(\frac{4-x}{2}\right) - \frac{2}{3} \times \frac{1}{2} \left(\frac{4-x}{2}\right)^2 \right] dx$$

Problem 20. (5 marks) Compute the volume of the sphere $S = \{(x,y,z) \mid x^2+y^2+z^2 \le 4\}$.
Hint: Use spherical coordinates (r,θ,ϕ) for which $x = r\sin\phi\cos\theta$, $y = r\sin\phi\sin\theta$, $z = r\cos\phi$ and $dx\,dy\,dz \simeq r^2\sin\phi\,dr\,d\theta\,d\phi$.

The integral is $\iiint_S dV$

and by using spherical coordinates this is

$$\iiint_{[0,2]\times[0,2\pi]\times[0,\pi]} r^2 \sin\phi \, dV'$$

and by Fubini this is

$$\int_0^{2\pi} \left(\iint_{[0,2]\times[0,\pi]} r^2 \sin\phi \, dA \right) d\theta$$

$$2\pi \int_0^2 \left(\int_0^\pi \sin\phi \, d\phi \right) r^2 \, dr$$

$$2\pi \left[-\cos\phi\right]_0^\pi \int_0^2 r^2 \, dr$$

$$2\pi \times 2 \times \frac{8}{3} = \frac{32\pi}{3}$$