Dawson College

Mathematics Department

Final Examination

Calculus III

201-BZF-05, Sections 01,02

May 24, 2019 14:00 pm – 17:00 pm

Instructors: **Richard Fournier, Nataliia Rossokhata**

Duration: **3 Hours**

Student Name ______________________________________

Student ID Number __________________________________

- Carefully read and fill out the cover sheet (name, ID number) and sign the integrity declaration.
- All questions are to be answered directly on the examination paper in the space provided.
- Solve the problems in the booklet provided clearly identifying each question and show all your work clearly.
- Only calculators Sharp EL531, X, XG and XT approved by department of mathematics are permitted.
- This examination consists of 20 problems.
- There are 11 pages including the cover page.
- Please ensure that you have a complete examination before starting.
- **This exam must be returned intact.**

Question	Marks
1	
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[5 marks] Question 1. Find a power series representation and its interval of convergence for $f(x) = \frac{1+x}{(1-x)^3}$.

$$\frac{1}{1-x} = \sum_{n \geq 0} x^n \Rightarrow \frac{x}{(1-x)^2} = x \frac{d}{dx} \frac{1}{(1-x)} = \sum_{n \geq 1} n x^n$$

$$\Rightarrow \frac{1+x}{(1-x)^3} = \frac{d}{dx} \frac{x}{(1-x)^2} = \sum_{n \geq 1} n^2 x^{n-1} = \sum_{n \geq 0} (n+1)^2 x^n$$

Clearly $I = (-1, 1)$

[5 marks] Question 2. Find the sum of the series $\sum_{n=1}^{\infty} \frac{n^2}{2^{n-1}}$.

According to Question 1

$$\sum_{n \geq 1} \frac{n^2}{2^{n-1}} = \sum_{n \geq 1} n^2 \left(\frac{1}{2}\right)^{n-1} = \frac{1 + 1/2}{(1 - 1/2)^3} = \frac{3/2}{1/8} = 12$$

[5 marks] Question 3. Approximate the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n+1)^4 n!}$ with an error at most $1/1500$.

The series is an alternating series and according to Leibniz, for any $k \geqslant 1$,

$$\left| \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n+1)^4 n!} - \sum_{n=1}^{k} \frac{(-1)^{n-1}}{(n+1)^4 n!} \right| \leqslant \frac{1}{(k+2)^4 (k+1)!}$$

with $(k+2)^4 (k+1)! = 3^4 \cdot 2! = 162 < 1500$

but $(k+2)^4 (k+1)! = 4^4\, 3! = 256 \times 6 > 1500$

The desired approximation can be chosen as $\sum_{n=1}^{2} \frac{(-1)^{n-1}}{(n+1)^4 n!} = \frac{1}{16} - \frac{1}{162} \simeq .0563271$

[5 marks] Question 4. Evaluate $\int_0^1 e^{-x^4} dx$ as an infinite series.

$e^x = \sum_{n \geqslant 0} \frac{x^n}{n!}$ with $R = \infty$

$e^{-x^4} = \sum_{n \geqslant 0} \frac{(-1)^n x^{4n}}{n!}$

$\int_0^1 e^{-x^4} dx = \sum_{n \geqslant 0} \frac{(-1)^n}{n!} \int_0^1 x^{4n} dx = \sum_{n \geqslant 0} \frac{(-1)^n}{n!(4n+1)}$

[5 marks] Question 5. Given the curve with parametric equations

$$x = t^2, \qquad y = t^4 + t, \qquad t \geq 1$$

prove that $\frac{d^2y}{dx^2} \leq 2$ at all points on the curve.

Clearly $y = (t^2)^2 + \sqrt{t^2} = x^2 + \sqrt{x}$ for $x \geq 1$

$$\frac{dy}{dx} = 2x + \frac{1}{2} x^{-1/2}$$

$$\frac{d^2y}{dx^2} = 2 - \frac{1}{4} x^{-3/2} \leq 2$$

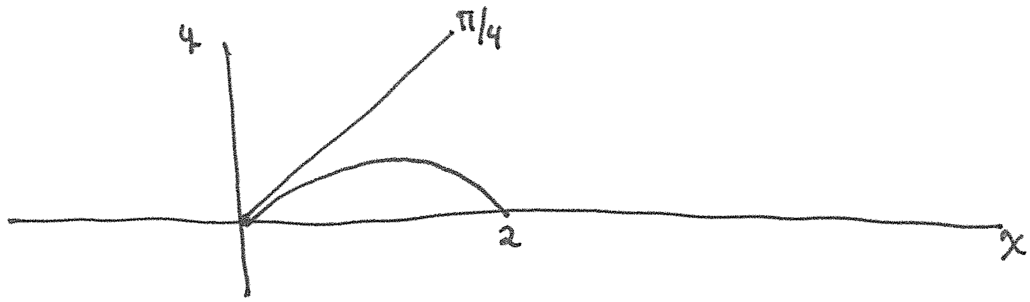
You may also use twice the rule

$$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$$

i.e. $$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dt} \Big/ \frac{dx}{dt}\right) \Big/ \frac{dx}{dt}$$

[5 marks] Question 6. Sketch the curve with polar equation $r = 2\cos(2\theta)$, $0 \leq \theta \leq \frac{\pi}{2}$.

Since this is a polar equation, $r \geq 0$ and in fact $r = 2\cos(2\theta)$, $0 \leq \theta \leq \frac{\pi}{4}$



[5 marks] Question 7. Find the length of the curve with parametric equations

$$x = \cos(\sqrt{t}), \qquad y = \sin(\sqrt{t}), \qquad 0 \le t \le \pi^2/4.$$

The curve is the same as $x = \cos(t)$, $y = \sin(t)$ with $0 \le t \le \pi/2$, i.e., a quarter of circle with radius 1 ; so $L = \pi/2$

Another trick!

$$L = \int_0^{\pi^2/4} \sqrt{x'(t)^2 + y'(t)^2}\, dt = \int_0^{\pi^2/4} \sqrt{\frac{\sin^2(\sqrt{t})}{4t} + \frac{\cos^2(\sqrt{t})}{4t}}\, dt$$

$$= \int_0^{\pi^2/4} \frac{1}{2\sqrt{t}}\, dt = \sqrt{t}\Big]_0^{\pi^2/4} = \frac{\pi}{2}$$

[5 marks] Question 8. Find the arc length parametrization for the helix with equation

$$x = 3\cos(t), \quad y = 3\sin(t), \quad z = 3t, \qquad 0 \le t \le 3.$$

$$s = \int_0^t \sqrt{x'(\ell)^2 + y'(\ell)^2}\, d\ell = \int_0^t \sqrt{9\sin^2(\ell) + 9\cos^2(\ell)}\, d\ell$$

$$= 3t$$

i.e $t = s/3$

and the requested parametrization is

$$x = 3\cos(s/3), \quad y = 3\sin(s/3), \quad 0 \le s \le 9$$

[5 marks] Question 9. Prove that the curve with the equation $\boldsymbol{r}(t) = (\cos t, \sin t, t^2)$, $t \geq 0$ does not have constant curvature $\kappa = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$.

$$r'(t) = (-\sin t, \cos t, 2t)$$

$$r''(t) = (-\cos t, -\sin t, 2)$$

$$\frac{|r'(t) \times r''(t)|}{|r'(t)|^3} = \frac{|(2\cos t + 2t\sin t, 2\sin t - 2t\cos t, 1)|}{|(\cos t, \sin t, 2t)|}$$

$$= \sqrt{\frac{4 + 4t^2 + 1}{4t^2 + 1}}$$

so that $K(0) = \sqrt{5}$ and $\lim_{t \to \infty} K(t) = 1 \neq \sqrt{5}$

and K is not constant

[5 marks] Question 10. Find an equation for the tangent line to the curve $\boldsymbol{r}(t) = (\frac{1}{t}, t^2, t^{-3})$ at the point (1,1,1).

$$r'(t) = \left(-\frac{1}{t^2}, 2t, -3t^{-4}\right)$$

$$r'(1) = (-1, 2, -3)$$

tangent line

$$l \longrightarrow l(-1, 2, -3) + (1, 1, 1), \quad -\infty < l < \infty$$

[5 marks] Question 11. Study the continuity of the function

$$f(x,y) = \begin{cases} \frac{xy\log(1+x^2y^2)}{x^2+y^2}, & if\ (x,y) \neq (0,0) \\ 0, & if\ (x,y) = (0,0) \end{cases}$$ at the point (0,0).

$$0 \leq |f(x,y)| = \frac{|xy|}{x^2+y^2} \log(1+x^2y^2) \leq \frac{1}{2} \log(1+x^2y^2)$$

It follows that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$

and f is cont. at $(0,0)$

[5 marks] Question 12. Find the maximum and minimum values of the function

$f(x,y,z) = x + y + z$ under the constraint $x^2 + y^2 + z^2 = 2$.

After Lagrange, there exists at any min. or max. point a multiplier L with

$$\nabla f + L\nabla g = (0,0,0) \quad \text{where } g(x,y,z) = x^2+y^2+z^2-2$$

i.e. $1+2Lx = 1+2Ly = 1+2Lz = 0$.

But since $L \neq 0$, we get $x = y = z$ and by the constraint $x = y = z = \sqrt{\frac{2}{3}}$ or $x = y = z = -\sqrt{\frac{2}{3}}$, i.e.,

$\max = 3\sqrt{\frac{2}{3}} = \sqrt{6}$ and $\min = -3\sqrt{\frac{2}{3}} = -\sqrt{6}$.

[5 marks] Question 13. Find all critical points of $f(x,y) = x^4 + y^4 - 2xy + 1$ and classify them.

The critical points are $(0,0)$, $\left(\frac{1}{\sqrt{2}}, +\frac{1}{\sqrt{2}}\right)$, $\left(-\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$

$(0,0)$ is a saddle point

the two other points are points of relative min.

(check with the standard criterium!)

[5 marks] Question 14. Prove or disprove: the function $f(x,y,z) = \sqrt[3]{xyz}$ is differentiable at (0,0,0).

Note that $\nabla f(0,0,0) = (0,0,0)$ since

$$\frac{\partial f}{\partial x}(0,0,0) = \lim_{h\to 0} \frac{f(h,0,0) - f(0,0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

$$= \frac{\partial f}{\partial y}(0,0,0) = \frac{\partial f}{\partial z}(0,0,0)$$

Therefore f is diff. at $(0,0,0)$ iff.

$$\lim_{(x,y,z)\to(0,0,0)} \frac{\sqrt[3]{xyz}}{\sqrt[2]{x^2+y^2+z^2}} = 0$$

but this is not the case since with $x=y=z\to 0$

the limit is

$$\lim_{x\to 0} \frac{\sqrt[3]{x^3}}{\sqrt[2]{3x^2}} = \lim_{x\to 0} \frac{x}{\sqrt[2]{3}\,|x|}$$ which does not exist

[5 marks] Question 15. Prove that the function $z = \cos(x + bt) - \sin(x - bt)$ is, for any real number b, a solution of

$$\frac{\partial^2 z}{\partial t^2} = b^2 \frac{\partial^2 z}{\partial x^2}.$$

$$\frac{\partial z}{\partial t} = -b\sin(x+bt) + b\cos(x-bt)$$

$$\frac{\partial^2 z}{\partial t^2} = -b^2\cos(x+bt) + b^2\sin(x-bt)$$

$$\frac{\partial z}{\partial x} = -\sin(x+bt) \cancel{*} \cos(x-bt)$$

$$\frac{\partial^2 z}{\partial x^2} = -\cos(x+bt) + \sin(x-bt) = \frac{1}{b^2}\frac{\partial^2 z}{\partial t^2}$$

[5 marks] Question 16. Find an equation for the tangent plane to the surface

$x^2 + y^2 + z^4 - 3x^5y^6z^7 = 0$ at the point $(1,1,1)$.

The normal of the tangent plane is $\nabla f(1,1,1)$

where $f(x,y,z) = x^2 + y^2 + z^4 - 3x^5y^6z^7$

i.e., $\nabla f(1,1,1) = (2-15,\ 2-18,\ 4-21)$

$= (-13, -16, -17)$

and the equation of the tangent plane is

$13(X-1) + 16(Y-1) + 17(Z-1) = 0$

$13X + 16Y + 17Z = 46$

[5marks] Question 17. Compute the double integral $\iint_R \; y\sin(xy)dA$ where $R = [1,2] \times [0,\pi]$.

By the Great Fubini, the integral is

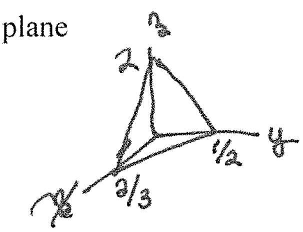
$$\int_0^{\pi} \left(\int_1^{2} y\sin(xy)\,dx \right) dy = \int_0^{\pi} -\cos(xy) \Big]_{x=1}^{x=2} dy$$

$$= \int_0^{\pi} \left[-\cos(2y) + \cos(y) \right] dy$$

$$= -2\sin(2y) + \sin(y) \Big]_{y=0}^{y=\pi}$$

$$= 0$$

[5marks] Question 18. Find the volume of the tetrahedron bounded by the plane $-z = -2 + 3x + 4y$ and the three coordinate planes.



By Fubini

$$V = \int_0^{2/3} \left(\int_0^{-\frac{3}{4}x + \frac{1}{2}} \left(\int_0^{2-3x-4y} 1 \, dz \right) dy \right) dx$$

$$= \frac{1}{9}$$

[5marks] Question 19. Find the volume of the solid that lies inside the sphere $x^2 + y^2 + z^2 = 4$ and inside the cylinder $x^2 + y^2 = 1$.

By Fubini

$$V = \iint_{D} 2\sqrt{4 - x^2 - y^2}\, dA \qquad D = \{(x,y) \mid x^2 + y^2 \le 1\}$$

$$= \iint_{D'} 2\sqrt{4 - r^2}\, r\, dA' \qquad D' = [0, 2\pi] \times [0, 1]$$

$$= 2 \int_0^{2\pi} \left(\int_0^1 \sqrt{4 - r^2}\, r\, dr \right) d\theta = 4\pi \int_0^1 \sqrt{4 - r^2}\, r\, dr$$

substitute $T = 4 - r^2$

[5marks] Question 20. Prove that $\iiint_E e^{-(x^2+y^2+z^2)^{3/2}}\, dV \le \frac{4\pi}{3}$ where

$E = \{(x, y, z) \mid x^2 + y^2 + z^2 \le 1\}$.

(**Hint:** you may use spherical coordinates $x = \rho \sin\varphi \cos\theta$, $y = \rho \sin\varphi \sin\theta$, $z = \rho \cos\varphi$ for which $dV = \rho^2 \sin\varphi\, d\rho d\varphi d\theta$.)

One simple solution, avoiding the hint :

$e^{-(x^2+y^2+z^2)^{3/2}} \le e^0 = 1$ and therefore

$$\iiint_E e^{-(x^2+y^2+z^2)^{3/2}}\, dV \le \iiint_E 1\, dV = \text{volume}(E) = \frac{4}{3}\pi(1)^3 = \frac{4}{3}\pi$$

Otherwise the integral equals, by Fubini

$$\int_0^{2\pi} \left(\int_0^{\pi} \sin(\varphi) \left(\int_0^1 e^{-\rho^3} \rho^2\, d\rho \right) d\varphi \right) d\theta \quad \ldots$$