Dawson College
Mathematics Department
Final Examination

Fall 2023 Semester: (Deferred until) Tuesday, January 9-th (14:00 pm to 17:00 pm)

201-BZS-05 (Sec. 00001), Probability and Statistics (Science)

Examiner: S. Shahabi.

Student's Full Name: ______________________ **ID:** ______________

- Print your name and student ID number in the space provided above;
- All questions are to be answered directly on the examination paper in the space provided. If you need more space for your answer, use the back of the page;
- No book, notes, graphing/programmable calculator or cellphones are permitted. You are only permitted to use the Sharp EL-531XG calculator during the examination;
- A Formula Sheet and the relevant Stat Tables are provided by the examiner;
- You must show all your work and justify all your answers;
- This examination booklet, all the Statistics Tables and the Formula Sheets must be returned intact.

Question	# Marks	student's scores
1	10	
2	5	
3	6	
4	5	
5	5	
6	5	
7	8	
8	6	
9	8	
10	6	
11	8	
12	8	
13	8	
14	5	
15	7	
Total	100	

THERE ARE 15 QUESTIONS IN TOTAL

THE EXAMINATION BOOKLET CONTAINS 8 PAGES
(INCLUDING THE COVER PAGE)

1. **[10 pts.]** How many strings of six (lowercase) **distinct** letters from the English alphabet contain:

(i) the letters a & b?

$\{a, b, \ , \ , \ , \ \}$ $\binom{24}{4} \cdot 6!$ [2] — — — — — —

(ii) the letters a & b in consecutive positions with a preceding b?

$\{ab, \ , \ , \ , \ \}$ $\binom{24}{4} \cdot 5!$ [2]

(iii) the letters a & b, where a is somewhere to the left of b?

the answer to (iii) is half the answer to (i) $= \frac{1}{2} \cdot \binom{24}{4} \cdot 6!$ [2]

(why?)

(iv) exactly two vowels? (the vowels: $\{a, e, i, o, u\}$)

$\binom{5}{2} \cdot \binom{21}{4} \cdot 6!$ [2]

(v) at least two vowels?

$\sum_{k=2}^{5} \binom{5}{k} \cdot \binom{21}{6-k} \cdot 6!$ [2]

which is the same as $P(26,6) - P(21,6) - \binom{5}{1} \cdot \binom{21}{5} \cdot 6!$

(no vowel) (one vowel)

2. **[5 pts.]** A newspaper advertises 5 different movies, 3 different plays and 2 tennis games. If a couple selects three activities *randomly*, what is the probability that they attend (exactly) two different types of activities?

Movie & play : $\binom{5}{2} \cdot \binom{3}{1} + \binom{5}{1} \cdot \binom{3}{2} = 45$ [1.5]

Movie & Tennis : $\binom{5}{2} \cdot \binom{2}{1} + \binom{5}{1} \binom{2}{2} = 25$ [1.5]

Play & Tennis : $\binom{3}{2} \cdot \binom{2}{1} + \binom{3}{1} \cdot \binom{2}{2} = 9$ [1.5]

$\Rightarrow P(\text{exactly 2 diff activ..}) = \frac{45 + 25 + 9}{\binom{10}{3}}$ [0.5]

$= \frac{79}{120} = 0.658\bar{3}$

3. **[6 pts.]** There are n married couples in a chess club. **(i)** If two club members are selected, *at random*, to play a match, what is the probability that they are married to each other?

[3] $$\frac{n}{\binom{2n}{2}} = \frac{n}{\frac{(2n)(2n-1)}{2}} = \frac{1}{2n-1}$$

(ii) What if the selection consists of choosing one man and one woman?

[3] $$\frac{n}{\binom{n}{1}\binom{n}{1}} = \frac{1}{n}$$

4. **[5 pts.]** In tossing a pair of *fair* dice, define the events: E = getting double, F = getting sum of 10.
(i) Evaluate $\Pr(E \cup F) = P(E) + P(F) - P(E \cap F)$ [1] [1] [1]

$$= \frac{6}{36} + \frac{3}{36} - \frac{1}{36} \qquad (E \cap F = \{(5,5)\})$$

$$= \frac{2}{9} = 0.\overline{2}$$

(ii) Decide, with reasoning, if E and F are independent or not.

$$P(E \cap F) = \frac{1}{36} \quad [2]$$

$$P(E) \cdot P(F) = \frac{6}{36} \cdot \frac{3}{36} \neq \frac{1}{36} \rightarrow \text{No!}$$

5. **[5 pts.]** Let p_n (for $n = 0, 1, 2, \ldots$) be the probability that a car policyholder will file for n insurance claims in a 5-year period. The actuary involved establishes the assumption that $p_{n+1} = (1/5)p_n$. What is the prob. that a randomly selected policyholder will file two or more claims during this period?

[1] [1]

$$\sum_{n=0}^{\infty} P_n = P_0 + P_1 + P_2 + P_3 + \cdots = 1 \rightarrow P_0 + \frac{1}{5}P_0 + \frac{1}{5^2}P_0 + \frac{1}{5^3}P_0 + \cdots = 1$$

[1] [1]

$$\rightarrow P_0\underbrace{\left(1 + \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \cdots\right)}_{\text{geom. series}} = 1 \rightarrow \frac{5}{4}P_0 = 1 \rightarrow P_0 = \frac{4}{5}$$

geom. series $= \frac{1}{1-\frac{1}{5}} = \frac{5}{4}$

[1]

$$\Rightarrow P(X \geq 2) = 1 - P_0 - P_1 = 1 - \frac{4}{5} - \frac{1}{5}\cdot\frac{4}{5} = \frac{1}{25}$$

6. **[5 pts.]** A sales firm receives, on the average, 24 calls per hour on its toll-free number. For any given half-an-hour period, find the probability that it will receive

$\lambda_{old} = 24 \Rightarrow \lambda_{new} = 12$ [1]

(i) at most 1 calls?

$$P(X \leq 1) = \sum_{k=0}^{1} \frac{e^{-\lambda} \cdot \lambda^k}{k!} = \frac{e^{-12} \cdot 12^0}{0!} + \frac{e^{-12} \cdot 12^1}{1!} = \frac{13}{e^{12}}$$ [2]

(Poisson)

$\approx 0.00079874\ldots$

(ii) 3 or more calls?

$$P(X \geq 3) = 1 - \sum_{k=0}^{2} \frac{e^{-\lambda} \cdot \lambda^k}{k!} = 1 - e^{-12}\left(1 + \frac{12}{1} + \frac{12^2}{2}\right)$$

$$= 1 - \frac{85}{e^{12}} \approx 0.99947742\ldots$$ [2]

7. **[8 pts.]** Suppose that 8% of all professional bicycle racers use steroids (for doping!), that a bicyclist who uses steroids tests positive for steroids 96% of the time, and that a bicyclist who does not use steroids tests positive for steroids 9% of the time. What is the probability that a randomly selected bicyclist who tests positive for steroids actually uses steroids?

U: used drug $+$: tested positive $-$: tested negative

Given: $P(U) = 0.08$, $P(+\mid U) = 0.96$, $P(+\mid U') = 0.09$ [2]

$\Rightarrow P(U') = 0.92$ [1]

We want $P(U \mid +) \overset{\text{(Bayes)}}{=} \dfrac{P(U) \cdot P(+\mid U)}{P(U) \cdot P(+\mid U) + P(U')P(+\mid U')}$ [3]

$$= \frac{(0.08)(0.96)}{(0.08)(0.96) + (0.92)\cdot(0.09)} = \frac{8 \times 96}{8 \times 96 + 9 \times 92} = \ldots$$ [2]

$$= \frac{768}{1596} = \frac{192}{399} \approx 0.48120300\ldots$$

8. **[6 pts.]** A coffee wending machine fills cups normally with a standard deviation of 0.3 oz. What should the mean be set at, so that 8 oz cups will overflow no more than 1% of the time?

(X: the amount of coffee poured by the machine)

$P(X > 8) = 0.01$ [1.5] $\Rightarrow P\left(Z > \frac{8-\mu}{0.3}\right) = 0.01$ [1.5]

$\xrightarrow{\text{z-table}} \frac{8-\mu}{0.3} = 2.325$ [1.5] $\Rightarrow \mu = 8 - (0.3)(2.325) \approx 7.3$ (oz) [1.5]

9. **[8 pts.]** The time needed for a student to complete a statistics exam is a normal dist. with $\mu = 160$ and $\sigma = 10$ (measurements are in minutes). What is the prob. that more than $(3/4)^{\text{th}}$ of a class of 28 students who write the exam will complete it, if it terminates after three hour?

X : normal

$p = P(\text{a rand. selec. student completes the exam})$ [2]

$= P(X \leq 180) = P\left(Z \leq \frac{180-160}{10}\right) = P(Z \leq 2) = 0.9772$

$\Rightarrow$ the theoretical answer $= P(B \geq 22)$ (B: binomial) $= \sum_{k=22}^{28} b(k; 28, p)$ [2] (binomial) $\begin{cases} \mu_1 = 27.3616 \\ \sigma_1 = 0.7898 \end{cases}$

[2]

$\approx P(N \geq 21.5) = P\left(Z \geq \frac{21.5 - 27.3616}{0.7898}\right)$

(N: (another) normal)

$\mu_2 = \mu_1 = 27.3616$

$\sigma_2 = \sigma_1 = 0.7898$

$= P(Z \geq -7.42)$ [2]

≈ 1

(almost certain!)

(Normal approximation of binomial)

10. **[6 pts.]** The mean lifetime of a product is 25 months, with a s.d. of 5 months. Assuming normality, for how many months should a guarantee be made if the manufacturer does not want to exchange more than 10% of the items?

X: the life span [2]

$P(X \le x_0^{?}) = 0.9 \Rightarrow P\left(Z \le \frac{x_0 - 25}{5}\right) = 0.9$ [2]

[1] [1]

z-table $\Longrightarrow \frac{x_0 - 25}{5} = 1.285 \Rightarrow x_0 = 31.425$

So, the manufacturer may make a guarantee of 31.4 months, if he/she does not want to exchange more than 10% of the items.

11. **[8 pts.]** A physician claims that joggers' maximal volume oxygen uptake is greater than the average of all adults, which is 36.7 (milliliter per kilogram). Is there sufficient evidence to support the physician's claim, at $\alpha = 0.05$, if a sample of 15 joggers has a mean of 40.6 (ml/kg) and a s.d. of 6 (ml/kg)?

$\begin{cases} H_0: \mu \le 36.7 \\ H_a: \mu > 36.7 \end{cases}$ [2.5] (μ = the average for the joggers (only)) (one-tailed)

$\bar{X} = 40.6 \quad n = 15$ (small sample) [2.5]

$t_0 = \frac{40.6 - 36.7}{6/\sqrt{15}} = 2.51 > t_\alpha = t_{0.05} = 1.761$ (d.f. = 14)

$\Rightarrow t_0 \in C.R. \Longrightarrow$ We may reject H_0 [1.5]

$\Rightarrow$ The physician is probably right. [1.5]

12. **[8 pts.]** A study found that 12 out of 34 Small Nursing Homes had a resident vaccination rate of less than 80%, while 17 out of 24 Large Nursing Homes had a resident vaccination rate of less than 80%. At $\alpha = 0.05$, test the claim that there is a significant difference in the proportions of the SNH's and LNH's with residence vaccination rate of less than 80%.

$$\begin{cases} H_0: P_1 - P_2 = 0 \\ H_a: P_1 - P_2 \neq 0 \end{cases} \quad [2.5] \qquad \hat{P}_1 = \frac{12}{34} \qquad \hat{P}_2 = \frac{17}{24} \qquad \hat{p} = \frac{12+17}{34+24} = \frac{29}{58} = 0.5 \quad [1]$$

[1]

[2]

$$Z_0 = \frac{\left(\frac{6}{17} - \frac{17}{24}\right) - (0)}{\sqrt{(0.5)(0.5)\left(\frac{1}{34} + \frac{1}{24}\right)}} = -2.67 > Z_{\alpha/2} = Z_{0.025} = 1.96$$

$\Rightarrow Z_0 \in$ C.R. $\longrightarrow$ We may reject Ho. [1.5]

Conclusion: there seems to be a significant difference.

13. **[6+2 pts.]** Consider the test scores for 10 students before and after a tutoring session:

Before	55	62	60	70	56	59	60	64	72	58
After	59	62	63	72	61	59	58	67	75	69
A − B:	4	0	3	2	5	0	−2	3	3	11

[1]

(i) Is there a significant improvement? Conduct a T. of H., using $\alpha = 0.05$;

$\bar{d} = 2.9 \quad S_d = 3.54$

$$\begin{cases} H_0: \mu_d \leq 0 \\ H_a: \mu_d > 0 \end{cases} \quad [1.5] \qquad [0.5] \qquad t_\alpha = t_{0.05} = 1.833 < t_0 = \frac{\bar{d} - \mu_d}{S_d/\sqrt{n}} = 2.59 \quad [1]$$

(d.f. = 9)

[1]

$\Rightarrow t_0 \in$ C.R. $\Longrightarrow$ we may reject Ho, hence accept Ha.

Thus, there is probably a significant improvement.

[1]

(ii) Construct a 90% C.I.E. for the mean improvement in session.

$$\bar{d} - t_{\alpha/2}\frac{S_d}{\sqrt{n}} < \mu_d < \bar{d} + t_{\alpha/2}\frac{S_d}{\sqrt{n}} \quad [1]$$

$$2.9 - (1.833)\frac{3.54}{\sqrt{10}} < \mu_d < 2.9 + (1.833)\frac{3.54}{\sqrt{10}} \quad [1]$$

$$0.85 < \mu_d < 4.95 \quad \text{or} \quad I_{90\%} = [0.85, 4.95]$$

14. **[5 pts.]** An ATM machine is available Monday through Friday only. The manager randomly selected one week and counted the number of people who used this ATM (as shown below) and conjectured that the number of people who use this ATM each of the five days of the week are **not** the same. At a 5% level of significance, test this conjecture.

Day	Mon.	Tue.	Wed.	Thu.	Fri.	
Number of Users	253	197	204	279	267	← O_i
	240	240	240	240	240	← E_i

{ H_0: the # of users are equal each day [1]
{ H_a: not as in H_0

$$\chi_0^2 = \sum_{i=1}^{5} \frac{(O_i - E_i)^2}{E_i} = \frac{13^2 + 43^2 + 36^2 + 39^2 + 27^2}{240} = 23.18\bar{3} \quad [2]$$

$$\chi_\alpha^2 = \chi_{0.05}^2 = 9.488 < \chi_0^2 \longrightarrow \chi_0^2 \in C.R. \quad [1]$$

(d.f. = 4)

→ We may reject H_0 → Conclusion: the manager's conjecture is probably correct.

[1]

15. **[7 pts.]** A sociologist wishes to see whether the number of years of education a person has completed is related to her/his place of residence. A sample of 88 people is selected (randomly) and classified as shown. At $\alpha = 0.05$, can the researcher conclude that a person's residential location is independent of the number of years of college?

Location	No college	Four-year degree	Advanced degree	
Urban	15, 11.53	12, 13.92	8, 9.55	35
Suburban	8, 10.55	15, 12.73	9, 8.73	32
Rural	6, 6.92	8, 8.35	7, 5.73	21
	29	35	24	88

[1]

{ H_0: the # of years of education is independent on the residence location [1]
{ H_a: not as in H_0

[3]

$$\chi_0^2 = \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{(O_{ij} - E_{ij})^2}{E_{ij}} = 3.006 < \chi_\alpha^2 = 9.488$$

(d.f. = 4)

[1]

⇒ We cannot reject H_0

⇒ Conclusion: the researcher is probably right.

[1]