Dawson College

Mathematics Department

Final Examination

Winter 2022: Wednesday, May 25-th (9:30 am to 12:30 pm)

201-BZS-05 (Sec. 00001,00002), Probability and Statistics (Science)

Examiner: S. Shahabi.

Student's Full Name: ______________________ ID: ____________

- Print your name and student ID number in the space provided above;
- All questions are to be answered directly on the examination paper in the space provided. If you need more space for your answer, use the back of the page;
- No book, notes, graphing/programmable calculator or cellphones are permitted. You are only permitted to use the Sharp EL-531XG calculator during the examination;
- A Formula Sheet and the relevant Stat Tables are provided by the examiner;
- You must show all your work and justify all your answers;
- This examination booklet, all the Statistics Tables and the Formula Sheets must be returned intact.

With Solutions

Question	# Marks	student's scores
1	5	
2	5	
3	5	
4	5	
5	8	
6	8	
7	8	
8	8	
9	8	
10	8	
11	8	
12	8	
13	8	
14	8	
Total	100	

THIS EXAMINATION BOOKLET CONTAINS 8 PAGES
(INCLUDING THIS COVER PAGE)

1. **[5 pts.]** How many permutations of the six letter "A, B, C, D, E, F" are there, **if** exactly three of the letters have been left in their original position? [One Example: "**A**, C, E, **D**, B, **F**", where only A, D and F are in their original places.]

- First we need to choose 3 letters, to be left in their original places: this is done $\binom{6}{3}$ ways.

- As for the other 3 letters, we need to find out in how many ways they are permuted with no fixed letter: This can be done in 2 ways (for "$\alpha\beta\gamma$", they are "$\beta\gamma\alpha$" and "$\gamma\alpha\beta$").

- The final answer to our problem then is: $\binom{6}{3} \cdot 2 = \underline{40}$.

2. **[5 pts.]** Suppose A, B and C are *independent* events. *Showing your work*, find $\mathbf{P}(A \cup B \cup C) + \mathbf{P}(A')\mathbf{P}(B')\mathbf{P}(C')$.

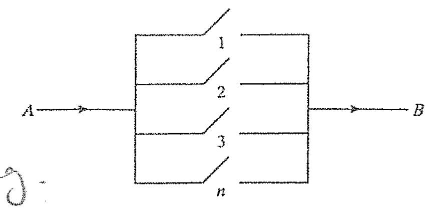
Rmk: If two events, say E & F, are indep., one knows that E' & F, E & F', and E' & F' are also indep..

And now

$$P(A \cup B \cup C) + P(A')P(B')P(C') \quad \text{indep.}$$
$$= P(A \cup B \cup C) + P(A' \cap B' \cap C')$$
$$= P(\underbrace{A \cup B \cup C}_{G}) + P((A \cup B \cup C)')$$
$$= P(G) + P(G') = \underline{1}.$$

(We are using $A' \cap B' \cap C' = (A \cup B \cup C)'$)

3. [**5 pts.**] A system composed of n separate components is said to be a parallel system if it functions when at least one of the components functions. If component i ($i = 1, ..., n$), which is *independent of the other components*, functions with probability p_i, express (in terms of the p_i's) the "probability that the system operates".



- E_i = the event that the i^{th} component is not working. $1 \le i \le n$

$P(E_i) = 1 - p_i$

- $E_1 \cap E_2 \cap \cdots \cap E_n$ = the event that no component is functioning.

= the system is not working

- $P(E_1 \cap \cdots \cap E_n) \overset{\text{indep.}}{=} P(E_1)P(E_2)\cdots P(E_n)$
$= (1-p_1)(1-p_2)\cdots(1-p_n)$

$\Rightarrow P(\text{system functioning}) = 1 - P(E_1 \cap \cdots \cap E_n)$
$= \boxed{1 - (1-p_1)(1-p_2)\cdots(1-p_n)}$

4. [**5 pts.**] The average number of phone inquiries **per day** at a poison control center (in a large hospital) is 7. Give the probability that the center will receive at least 50 calls in *one week.*

- the average # of calls per week = $(7)(7) = 49$

- this is a Poisson dist. with $\lambda = 49$

- We want $P(X \geq 50) = \sum_{n=50}^{\infty} \frac{e^{-\lambda}\lambda^n}{n!}$

$= 1 - \sum_{n=0}^{49} \frac{e^{-49}\, 49^n}{n!}$

$= 1 - e^{-49} \cdot \sum_{n=0}^{49} \frac{49^n}{n!}$

5. **[8 pts.]** The lifetime T (in years) of a certain brand of light-bulb is a cont. r.v. with the density fun. $f(t) = \frac{1}{10}e^{-t/10}$, $t \geq 0$. If 6 of these light-bulbs are installed in different places, what is the probability that at least 4 of them are still functioning after 12 years? (**Hint:** What is the probability that a **single** light-bulb lasts longer than 12 years?)

X = the # of light-bulbs still working after 12 years

We want $P(4 \leq X \leq 6) = \sum_{x=4}^{6} \binom{6}{x} p^x (1-p)^{6-x}$

where $p = P(T \geq 12)$ (with T as in the problem)

$$p = \int_{12}^{+\infty} \frac{1}{10} e^{-t/10}\, dt = \lim_{A \to +\infty} \left[-e^{-t/10} \right]_{12}^{A} = \lim_{A \to +\infty} \left(-e^{-A/10} + e^{-12/10} \right)$$

$$= \frac{1}{e^{1.2}} \approx 0.301194 \Rightarrow \boxed{\text{the answer} = \sum_{x=4}^{6} \binom{6}{x} \frac{1}{e^{1.2x}} \left(1 - \frac{1}{e^{1.2}}\right)^{6-x} \approx 0.07142}$$

6. A pair of *fair die* are rolled until a sum of "11" is observed *for the second time*; we then stop.

(a) [3 pts.] If X is the r.v. whose value represents the number of rolling when we stop, give a formula for the prob. dist. fun. $f(n) = \mathbf{P}(X = n)$, for $n = 2, 3, 4, 5, \ldots$

$P(\text{Sum} = 11) = \frac{2}{36} = 1/18$

$\underbrace{\bigcirc\bigcirc \cdots \bigcirc}_{\text{once : sum} = 11}$ (Sum 11) — n rolling

$$f(n) = \binom{n-1}{1} \frac{1}{18} \cdot \left(\frac{17}{18}\right)^{n-2} \frac{1}{18} = \frac{17^{n-2}(n-1)}{18^n} \quad (n = 2, 3, 4, \ldots)$$

(b) [5 pts.] *Verify* that $\sum_n f(n) = 1$. (**Hint:** A useful formula: $1 + 2x + 3x^2 + 4x^3 + \cdots = \frac{1}{(1-x)^2}$)

$$\sum_n f(n) = \sum_{n=2}^{\infty} \frac{17^{n-2}(n-1)}{18^n} = \sum_{m=0}^{\infty} \frac{17^m \cdot (m+1)}{18^{m+2}} \quad (m = n-2)$$

using with $x = \frac{17}{18}$

$$= \frac{1}{18^2} \sum_{m=0}^{\infty} (m+1) \left(\frac{17}{18}\right)^m = \frac{1}{18^2} \cdot \frac{1}{\left(1 - \frac{17}{18}\right)^2}$$

$$= \frac{1}{18^2} \cdot \frac{18^2}{1} = 1, \text{ as was to be shown!}$$

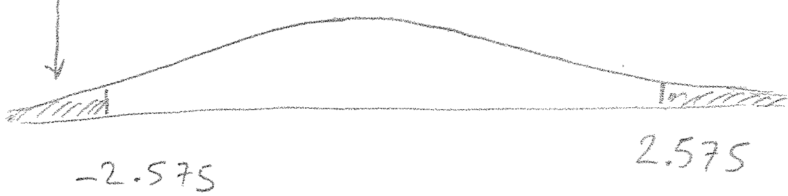
7. **[8 pts.]** A manufacturer of fishing-equipments has developed a new synthetic fishing line that s/he claims has a mean breaking strength of 8 kilograms with a s.d. of 0.5 kg. A random sample of 50 such lines is tested and found to have a mean breaking strength of 7.8 kg. Use $\alpha = 0.01$ to conduct a Test of Hypothesis to see if the sample supports the manufacturer's claim.

$\boxed{H_0: \mu = 8 \quad H_a: \mu \neq 8}$ (two-tailed test)

$\alpha = 0.01 \quad n = 50 \quad \bar{X} = 7.8 \qquad z_{\alpha/2} = 2.575$

$$z_0 = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} = \frac{7.8 - 8}{0.5/\sqrt{50}} = -2\sqrt{2} \approx -2.82$$

Since $z_0 < -z_{\alpha/2} \Rightarrow$ We may reject H_0,



Conclusion: the sample does not seem to support the manufacturer's claim.

8. **[8 pts.]** The blood sugar readings for 12 people with diabetes were recorded before and after a medication regime:

Before	128	145	115	156	129	147	130	127	120	145	124	135
After	117	133	108	133	111	129	122	124	103	129	116	117
Difference	11	12	7	23	18	18	8	3	17	16	8	18

Can we conclude that the medication lowers blood sugar by more than 10 points? Conduct a T. of H., using $\alpha = 0.05$.

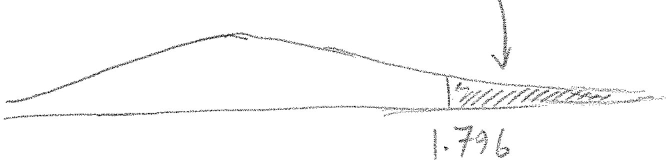
$\boxed{H_0: \mu_B - \mu_A \leq 10 \quad H_a: \mu_B - \mu_A > 10}$ $\quad \bar{d} = 13.25 \quad S_d = 5.9562$

a right-tailed test

$n = 12 \quad \alpha = 0.05$

$$t_0 = \frac{\bar{d} - 10}{S_d/\sqrt{n}} = 1.89 \qquad t_\alpha = 1.796 \ (\text{d.f.} = 11)$$

Since $t_0 > t_\alpha \Rightarrow$ We may reject H_0



Conclusion: the medication seems to have lowered blood sugar by more than 10 points.

9. **[8 pts.]** A vote is to be taken among the residents of a town and the surrounding county to determine whether a proposed chemical plant should be constructed. If 120 of 200 town voters favour the proposal and 240 of 500 county residents favour it, use $\alpha = 0.05$ to test the conjecture that the proportion of town voters favouring the proposal is higher than the proportion of county voters.

$\boxed{H_0: P_T \le P_C \qquad H_a: P_T > P_C}$

(right-tailed)

$\hat{P}_T = \frac{120}{200} \quad \hat{P}_C = \frac{240}{500} \quad n_1 = 200 \quad n_2 = 500$

$\hat{p} = \frac{120+240}{200+500} = \frac{18}{35} \quad \hat{q} = \frac{17}{35}$

0.514

$$Z_0 = \frac{(\hat{P}_T - \hat{P}_C) - (P_T - P_C)}{\sqrt{\hat{p}\hat{q}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = 2.8617$$

$Z_\alpha = 1.645$

Since $Z_0 > Z_\alpha \Rightarrow$ we may reject Ho.

Conclusion: the proportion of town voters favouring the proposal seems to be higher than the proportion of county voters.

10. **[8 pts.]** Use $\alpha = 0.05$ and the data below to test whether there is a significant difference in the average grades between two sections of a university (graduate) course? *Assume that the standard deviations of the two populations are equal.*

$\boxed{H_0: \mu_1 = \mu_2 \qquad H_a: \mu_1 \neq \mu_2}$

(two-tailed)

$\alpha = 0.05$

	Section 1	Section 2
n	7	13
$\sum x$	514	1062
$\sum x^2$	37867	86900

$\bar{X}_1 = 73.4 \quad S_1 = 4.56$

$\bar{X}_2 = 81.7 \quad S_2 = 3.45$

$S_p = 3.86$

$$t_0 = \frac{(73.4 - 81.7) - (0)}{3.86\sqrt{\frac{1}{7} + \frac{1}{13}}} = -4.59$$

$t_{\alpha/2} = 2.101$

(d.f. = 18)

Since $t_0 < -2.101 \Rightarrow$ we may reject Ho.

Conclusion: there seems to be a significant difference in the average grades b/w the two sections.

11. **[8 pts.]** The test score of a student taking my (= Shahabi's) exam is (approximately) a normal random variable with mean 70, and standard deviation 10. How many students would have to take my exam to ensure, with probability at least 0.9, that the class average would be within 3 of 70?

$\Downarrow$

$\alpha \leq 0.1$

$\Downarrow$

$z_{\alpha/2} \geq z_{0.05} = 1.645$

$E \leq 3 \Rightarrow z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}} \leq 3$

$\Rightarrow \frac{10}{\sqrt{n}} \leq \frac{3}{1.645}$

$\Rightarrow \sqrt{n} \geq 5.48\overline{3}$

$\Rightarrow n \geq 30.066$

$\Rightarrow \boxed{n \geq 31}$

12. **[8 pts.]** A vending machine is designed to discharge exactly 9 ounces of coffee into a cup. To test whether the machine is operating properly, 25 cups were selected at random, yielding $\bar{x} = 8.6$ oz, and $S = 0.7$ oz. Conduct a Test of Hypothesis, using $\alpha = 0.05$, to see if the machine operates properly.

H_0: The machine operates as planned $\quad H_a$: Not as in H_0

$(H_0: \mu = 9 \quad H_a: \mu \neq 9)$

$t_0 = \frac{\overline{X}_0 - \mu}{S/\sqrt{n}} = \frac{8.6 - 9}{0.7/\sqrt{25}} = -2.8571$

$t_{\alpha/2} = 2.064$ (a two-tailed test)

(d.f. = 24) $\quad$ Since $t_0 < -t_{\alpha/2} \Rightarrow t_0 \in$ C.R.

Conclusion: the machine is probably not operating properly!

13. **[8 pts.]** A pair of die are rolled 360 times. Do the following results support the conjecture that the die are *both* fair? Test with $\alpha = 0.05$.

Sum	2	3	4	5	6	7	8	9	10	11	12
O_i	8	19	28	43	50	64	46	37	32	21	12
E_i	10	20	30	40	50	60	50	40	30	20	10

$\begin{cases} H_0: \text{both die are fair} \\ H_a: \text{not as in } H_0 \end{cases}$

$$\chi_0^2 = \frac{(8-10)^2}{10} + \frac{(19-20)^2}{20} + \frac{(28-30)^2}{30} + \frac{(43-40)^2}{40} + \frac{(50-52)^2}{50} + \frac{(64-60)^2}{60} + \frac{(46-50)^2}{50} + \frac{(37-40)^2}{40} + \frac{(32-30)^2}{30} + \frac{(21-20)^2}{20} + \frac{(12-10)^2}{10}$$

$= 2.28\bar{3}$

$\chi_\alpha^2 = 18.307$ (d.f. = 10)

Since $\chi_0^2 < \chi_\alpha^2$

we don't reject H_0. $\Longrightarrow$ $\boxed{\text{Conclusion: the die are (probably) both fair.}}$

14. **[8 pts.]** Using the results of a political poll shown below, test whether age and party affiliation are *independent*. Use $\alpha = 0.05$.

	Liberals	Conservatives	NDP	Sum
age < 25	18, 21	36, 44	46, 35	100
$25 \leq$ age ≤ 40	22, 25.2	48, 52.8	50, 42	120
age > 40	44, 37.8	92, 79.2	44, 63	180
Sum	84	176	140	400

$\begin{cases} H_0: \text{age \& party affiliation are indep.} \\ H_a: \text{not as in } H_0 \end{cases}$

$$\chi_0^2 = \sum\sum \frac{(O_{ij} - E_{ij})^2}{E_{ij}} = 16.523$$

$\chi_\alpha^2 = 9.488$ $\big((2)(2) = 4\big)$

$\chi_0^2 > \chi_\alpha^2 \Longrightarrow \chi_0^2 \in C.R.$

$\boxed{\text{Conclusion: Age \& party affiliation are \underline{not} independent!}}$