Dawson College

Mathematics Department

Final Examination
Winter - 2024

Differential Calculus 201-MA1-DW (201-103-DW) Social Science

Wednesday, May 29, 2024

14:00 – 17:00

Student Name: Solutions

Student I.D. #: ______________________________

Instructor Name: ______________________________

Instructors: **O. Veres, A. Gambioli, I. Rajput**

INSTRUCTIONS:

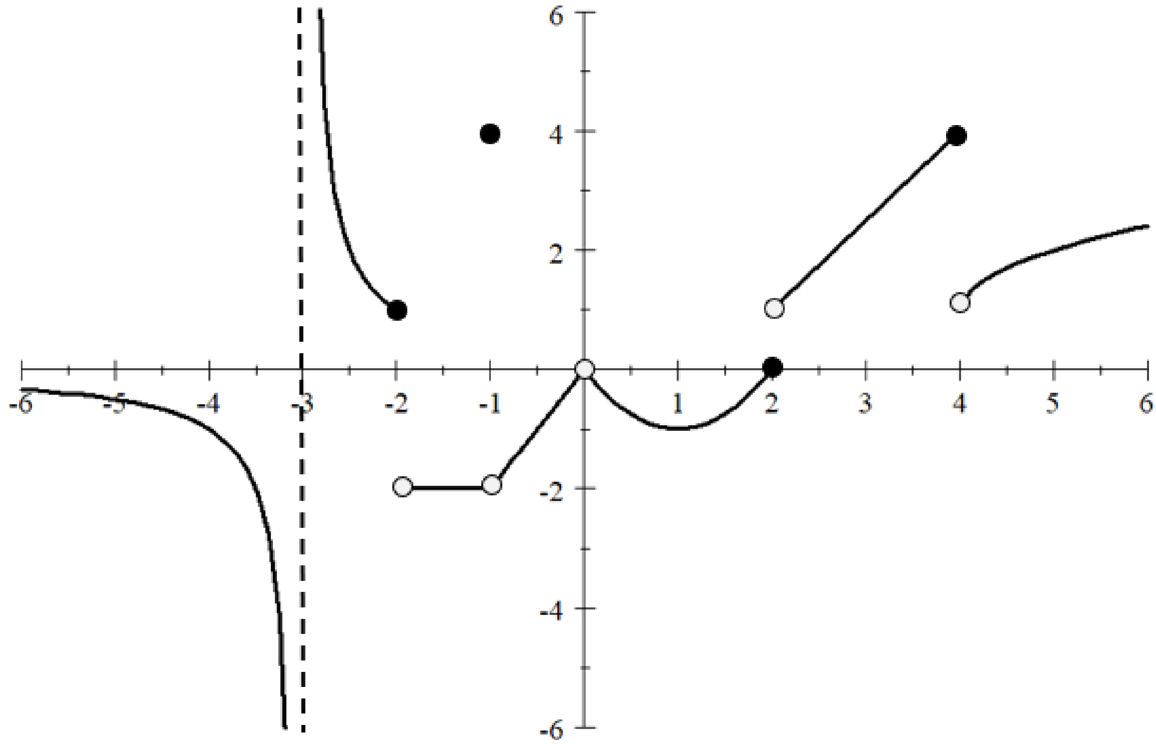
- Print your name, ID number and instructor's name in the space provided above.
- All questions are to be answered in the space provided for each question.
- Give detailed solutions and explain your work clearly.
- Reverse sides of pages may be used for rough work and/or to complete a solution.
- Only calculator permitted is SHARP EL-531***.
- This examination consists of 15 questions page 2 through 12. **Please ensure that you have a complete examination before starting.**

Reserved for Instructor

Question	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	
Marks	5	10	4	6	6	6	5	5	6	5	5	5	5	6	6	6	9	**=100**
Score																		

This examination booklet must be returned intact.

Q-1) [5-Marks] The graph of $y=f(x)$ is given below.



Use the above graph to answer the following questions. Use DNE if the limit does not exist and $-\infty$ or ∞ if needed.

a. $\lim_{x\to -2^-} f(x) = 1$

b. $\lim_{x\to -1} f(x) = -2$

c. $\lim_{x\to 2} f(x) =$ DNE

d. $\lim_{x\to -\infty} f(x) = 0$

e. $\lim_{x\to -3^-} f(x) = -\infty$

f. $\lim_{x\to 0} f(x) = 0$

g. $f(-1) = -4$

h. $f(4) = 4$

i. State the condition of continuity that is violated at $x=-1$.
$f(-1) \neq \lim_{x\to -1} f(-1)$, $[condition\ 3]$

j. Write an equation of the vertical asymptote (if any): $x=-3$

Q-2) [10 marks] Use algebraic techniques to evaluate the following limits, if they exist; show your work and write $-\infty$ or ∞ if needed.

a.

$$\lim_{x\to -1} \frac{\frac{x+2}{x}+1}{x+1} \quad \left(\frac{0}{0}\right)$$

$$\lim_{x\to -1} \frac{\frac{x+2+x}{x}}{x+1} = \lim_{x\to -1} \frac{\frac{2x+2}{x}}{x+1} = \lim_{x\to -1} \frac{2(x+1)}{x(x+1)}$$

$$= \lim_{x\to -1} \frac{2}{x} = -2$$

b.

$$\lim_{x\to 3} \frac{x-3}{\sqrt{5x+1}-4} \quad \left(\frac{0}{0}\right)$$

$$\lim_{x\to 3} \frac{x-3}{\sqrt{5x+1}-4} \times \frac{\sqrt{5x+1}+4}{\sqrt{5x+1}+4} = \lim_{x\to 3} \frac{(x-3)\left(\sqrt{5x+1}+4\right)}{(\sqrt{5x+1})^2-(4)^2}$$

$$\lim_{x\to 3} \frac{(x-3)\left(\sqrt{5x+1}+4\right)}{5(x-3)} = \lim_{x\to 3} \frac{\sqrt{5x+1}+4}{5} = \frac{8}{5}$$

Q-3) [4 marks] It is estimated that the average cost of producing x units of an auto-part is given by $\overline{C}(x)=\dfrac{175x+195000}{x}$. Evaluate $\lim\limits_{x\to\infty}\overline{C}(x)$ and interpret your result. Show all the steps to reach to your conclusion.

$$\overline{C}(x)=\lim_{x\to\infty}\frac{\frac{175x}{x}+\frac{195000}{x}}{\frac{x}{x}}=\lim_{x\to\infty}\frac{175+\frac{195000}{x}}{1}=175$$

Average cost in the long run is reaching to \$175

Q-4) The function that calculates the provincial income tax for a married couple with two dependant children is given by.

$$D(x)=\begin{cases}0.12794x-638.7 & \text{if} \quad 0<x\le 30000\\ 3200+0.155(x-30000) & \text{if} \quad 30000<x\le 45000\\ 5525+0.185(x-45000) & \text{if} \quad 45000<x\le 60000\\ 8300+0.23(x-60000) & \text{if} \quad 60000<x\le 85000\\ 13525+0.275(x-85000) & \text{if} \quad x>85000\end{cases},$$

where x is the yearly income.

a. [4 marks] Use the **three** conditions of continuity to determine if the function is continuous at $x=60000$.

1) $D(60000)=5525+0.185(60000-45000)=8300$ is defined

2) $\lim\limits_{x\to 60000^+}\left(8300-0.23(60000-60000)\right)=8300$

$\lim\limits_{x\to 60000^-}\left(5525-0.185(60000-45000)\right)=8300$

$\to \lim\limits_{x\to 60000}D(x)=8300$

3) $D(60000)=\lim\limits_{x\to 60000}D(x)=8300, \quad \textit{continious at} \;\; x=60000$

b. [2 marks] How much tax needs to be paid if the couple earns \$355,685.50?

$$D(355685.50)=13525+0.275(355685.50-85000)=\$87,963.51$$

Q-5) [6 marks] Use the limit definition of the derivative (four-step process) to find $f'(x)$ of the function

$$f(x) = 3 - 2x + 3x^2$$

$$\begin{aligned} f(x+h) &= 3 - 2(x+h) + 3(x+h)^2 \\ &= 3 - 2x - 2h + 3x^2 + 6xh + 3h^2 \end{aligned}$$

$$\begin{aligned} f(x+h) - f(x) &= 3 - 2x - 2h + 3x^2 + 6xh + 3h^2 - 3 + 2x - 3x^2 \\ &= -2h + 6xh + 3h^2 \end{aligned}$$

$$\frac{f(x+h) - f(x)}{h} = \frac{-2h + 6xh + 3h^2}{h}$$

$$= \frac{h(-2 + 6x + 3h)}{h} = -2 + 6x + 3h$$

$$\begin{aligned} \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \to 0} (-2 + 6x + 3h) \\ &= -2 + 6x = f'(x) \end{aligned}$$

[Note: In Q-6, replace the required values without simplifying the derivatives to get answer]

Q-6) a. [3 marks] The number of electronic devices in use in North America (in millions) in year t from 2017 through 2019 is approximately $E(t)=134.2t^{1.85} + 231.7t + 2.75$, where $t=1$ corresponds to 2017. How fast was the use of devices changing in 2018?

$$E'(t)=134.2\left(1.85t^{0.85}\right)+231.7$$

$t = 2$, corresponds to 2018

$$E'(2)=134.2(1.85)\left(2^{0.85}\right)+231.7=679.21$$

b. [3 Marks] The revenue function of a product is given by $R(x)=175x-0.05x^2\ln x$. Find the marginal revenue when $x=150$.

$$R'(x)=175-0.05\left[2x\ln x+\frac{x^2}{x}\right]$$
$$=175-0.1x\ln x-0.05x$$

$$R'(150)=175-0.1(150)\ln 150-0.05(150)$$
$$=175-75.1595-7.5=92.34\$$$

Q-7) [5 marks] The quantity sold S of a new product is given as a function of time t, where t is measured in months.

$$S(t) = \frac{100,000t^2}{(2t+1)^2} \qquad t > 0$$

Find $S'(36)$. Is the quantity increasing or decreasing?

$$S'(t) = \frac{(2t+1)^2 200000t - 100000t^2 \cdot 2(2t+1)(2)}{(2t+1)^4}$$

$$S'(t) = \frac{200000t(2t+1)[2t+1-2t]}{(2t+1)^4} = \frac{200000t(2t+1)}{(2t+1)^4}$$

$$S'(t) = \frac{200000t}{(2t+1)^3} \quad \to \quad S'(36) = \frac{7200000}{389017} \approx 18.51$$

Increasing at the rate of 18.51 per month

Q-8) [5 marks] It is projected that *t* years from now, the population of a certain town will be approximately P(t) thousand people, where

$$P(t) = \frac{80}{1 + e^{-0.1t}}.$$

At what rate will the population change 20 years from now?

$$P(t) = 80\left(1+e^{-0.1t}\right)^{-1}$$

$$P'(t) = 80(-1)\left(1+e^{-0.1t}\right)^{-2} e^{-0.1t}(-0.1) = \frac{8}{\left(1+e^{-0.1t}\right)^2 e^{0.1t}}$$

$$P'(20) = \frac{8}{\left(1+e^{-2}\right)^2 e^2} \approx 0.83994 \to \approx 840 \text{ people per year}$$

Q-9) [6 marks] Differentiate the following functions. *DO NOT SIMPLIFY.*

a. $y = \sqrt[3]{\ln(\sin 2x) + e^{-x} + 2}$

$$y' = \frac{1}{3}\left(\ln(\sin 2x) + e^{-x} + 2\right)^{-2/3}\left[\frac{2\cos 2x}{\sin 2x} - e^{-x}\right]$$

b. $y = \dfrac{3 + \sqrt{\tan x}}{\cos\left(x^2 + x\right)}$

$$y' = \frac{\cos\left(x^2+x\right)\left(\frac{1}{2}\tan^{-1/2} x \cdot \sec^2 x\right) - \left(3 + \sqrt{\tan x}\right)\left[-\sin\left(x^2+x\right)\cdot(2x+1)\right]}{\cos^2\left(x^2+x\right)}$$

Q-10) [5 marks] Given $f(x) = 3\ln(2x-1) + 9$, find $f''(x)$, and evaluate its value at $x = 1$.

$$f'(x) = \frac{6}{2x-1},$$

$$f''(x) = \frac{(2x-1)(0) - 6(2)}{(2x-1)^2}$$

$$f''(x) = \frac{-12}{(2x-1)^2}$$

$$f''(1) = \frac{-12}{1} = -12$$

Q-11) [5 marks] Use Logarithmic differentiation to find y' of the function $y=(3x+2)^{\sin x}$

$$\ln y=\ln(3x+2)^{\sin x} \quad\rightarrow\quad \ln y=\sin x\cdot\ln(3x+2)$$

$$\frac{y'}{y}=\cos x\cdot\ln(3x+2)+\frac{3\sin x}{3x+2}$$

$$y'=\left[\cos x\cdot\ln(3x+2)+\frac{3\sin x}{3x+2}\right]y$$

$$y'=\left[\cos x\cdot\ln(3x+2)+\frac{3\sin x}{3x+2}\right](3x+2)^{\sin x}$$

Q-12) The curve $y=f(x)$ is defined by the equation $(2+x)\,y^2=x^3+y+1$

a. [3 marks] Find $\frac{dy}{dx}$ using implicit differentiation.

$$y^2+(2+x)2yy'=3x^2+y'$$

$$(2+x)2yy'-y'=3x^2-y^2$$

$$y'[4y+2xy-1]=3x^2-y^2$$

$$y'=\frac{3x^2-y^2}{4y+2xy-1}$$

b. [2 marks] Find an equation of the tangent line at (1,1) on the curve defined above.

$$m=y'_{(1,1)}=\frac{2}{5};\quad (1\,,1)$$

$$y-1=\frac{2}{5}(x-1)\ \rightarrow\ y-1=\frac{2}{5}x-\frac{2}{5}\ \rightarrow\ y=\frac{2}{5}x+\frac{3}{5}$$

Q-13) [5 marks] The cost to produce x units per week is given by $C(x) = 0.1x^2 + 8000$. How fast is cost per week changing when production is changing at the rate of 10 units per week and the cost is $\$9000$.

$$\frac{dC}{dt} = ?, \quad \frac{dx}{dt} = 10 \text{ units/week}, \quad \mathrm{C}(x) = 0.1x^2 + 8000$$

$$9000 = 0.1x^2 + 8000 \ \rightarrow \ x^2 = \frac{1000}{0.1} \ \rightarrow x = 100 \text{ units}$$

$$\frac{dC}{dt} = 0.1\left(2x\frac{dx}{dt}\right) \quad \rightarrow \quad \frac{dC}{dt} = 0.1(2)(100)(10) = 200\$ / week$$

Q-14) [6 marks] For a dosage of x cubic centimeters (cc) of a certain drug, the resulting increase is systolic blood pressure B (in mmHg) is approximated by

$$B(x) = 3050x^2 - 18300x^3 \quad (0 \le x \le 0.16).$$

Find the relative extrema of the function $B(x)$ on the interval $(0\ ,\ 0.16)$, and the dosage x at which they occur, and classify them using the Second Derivative Test.

$$B'(x) = 6100x - 54900x^2 = x(6100 - 54900x)$$

$$B'(x) = 0, \quad x = 0, \ and \ \ x = \frac{6100}{54900} \approx 0.111\ cc$$

$$B''(x) = 6100 - 109800x$$

$$B''(0.111) = 6100 - 109800(0.11) = -6087.8 < 0$$

$$\text{maximum when } x \approx 0.111$$

$$B(0.111) = 3050(0.111)^2 - 18300(0.111)^3$$

$$\approx 12.551 \ \ mmHg$$

Q-15) [6 marks] A company can produce a maximum of 35000 widgets in a month. The demand function for these widgets is $p = 5 - 0.00025x \quad (0 \le x \le 35000)$, where p denotes the unit price in dollars and x the number of demanded units. The monthly cost function to produce these widgets is $C(x) = 500 + 3x - 0.0002x^2 \quad (0 \le x \le 35000)$. How many widgets should they try to sell in a month to maximize their profit? What is the maximal profit?

$$R(x) = xp = x(5 - 0.00025x) = 5x - 0.00025x^2, \qquad C(x) = 500 + 3x - 0.0002x^2$$

$$P(x) = R(x) - C(x) = 5x - 0.00025x^2 - 500 - 3x + 0.0002x^2$$

$$P(x) = -0.00005x^2 + 2x - 500$$

$$P'(x) = -0.0001x + 2 \quad \rightarrow \quad P'(x) = 0 \quad when \quad x = 20000$$

x	=	0	20000	35000
$P(x)$	=	-500	19500	8250

Maximum \$19,500 when $x = 20000$

Q-16) [6 marks] A cardboard box with a square base is to have a volume of $8000\,cm^3$. The cardboard for the box costs 0.1¢/cm^2, except the cardboard for the bottom, which is thicker, so it costs three times as much. Find the dimensions that will minimize the cost of the cardboard. Round the dimensions of the box to three decimal places.

$$V = x^2 h = 8000\,cm^3 \;\rightarrow\; h = \frac{8000}{x^2}\,cm$$

$$C(x) = 0.3x^2 + 0.1x^2 + 0.1(4xh)$$

$$C(x) = 0.3x^2 + 0.1x^2 + 0.4(x)\left(\frac{8000}{x^2}\right)$$

$$C(x) = 0.4x^2 + \frac{3200}{x}$$

$$C'(x) = 0.8x - \frac{3200}{x^2} = \frac{0.8x^3 - 3200}{x^2} = 0$$

$$0.8x^3 - 3200 = 0 \;\rightarrow\; x^3 = \frac{3200}{0.8} = 4000 \;\rightarrow\; x = \sqrt[3]{4000} \approx 15.874\ cm$$

$$h = \frac{8000}{(15.874)^2} \approx 31.748\ cm,$$

$$C''(x) > 0, Minimum \;\; when \;\; 15.874 \times 15.874 \times 31.748 \;\; cm$$

Q-17) Consider the function $f(x)=\frac{20x-40}{x^3}$, with its derivatives $f'(x)=\frac{120-40x}{x^4}$ and $f''(x)=\frac{120x-480}{x^5}$

(a) [1 mark] Find its x- and y- intercepts.

$$y-\text{int.}: f(0)\ \textit{is undefined},$$

$$x-\text{int.}\ \frac{20x-40}{x^3}=0$$

$$\rightarrow\ 20x-40=0 \quad \rightarrow \quad x=2$$

(b) [1 mark] Find its asymptotes, if any. Justify your answer using limits.

$$\lim_{x\to\pm\infty}\frac{20x-40}{x^3}=\lim_{x\to\pm\infty}\frac{\frac{20}{x^2}-\frac{40}{x^3}}{1}=\frac{0}{1} \qquad \rightarrow \quad y=0\ \ H.A.$$

$$\lim_{x\to 0^+}\frac{20x-40}{x^3}=-\infty, \qquad \lim_{x\to 0^-}\frac{20x-40}{x^3}=\infty \quad \rightarrow \quad x=0 \quad V.A.$$

(c) [2 marks] Find the intervals where f is increasing and where f is decreasing, and local extrema, if any.

$$f'(x) = 0 \quad when \quad x = 3, \quad undefined \ when \quad x = 0$$

$(-\infty, 0)$	$(0, 3)$	$(3, \infty)$
$f'(x) > 0$	$f'(x) > 0$	$f'(x) < 0$
$Inc.$	$Inc.$	$Dec.$

$$f(3) = \frac{20}{27} \ (\approx 0.74) \quad \text{Rel. Maximum}$$

(d) [2 marks] Find the intervals where f is concave upwards and where f is concave downwards, and point(s) of inflection, if any.

$$f''(x) = 0 \quad when \quad x = 4, \quad undefined \quad when \quad x = 0$$

$$f'(x) = 0 \quad when \quad x = 3, \quad undefined \ when \quad x = 0$$

$(-\infty, 0)$	$(0, 4)$	$(4, \infty)$
$f''(x) > 0$	$f''(x) < 0$	$f''(x) > 0$
$C.U$	$C.D$	$C.U$

$$f(4) = \frac{80 - 40}{64} = \frac{40}{64} = \frac{5}{8} \ (\approx 0.625)$$

$$\left(4 \ , \ \frac{5}{8}\right) \text{ Inflection Point}$$

(e) [3 marks] Graph the function and label the important points.