Dawson College

Mathematics Department

Final Examination

Winter - 2025

Differential Calculus 201-MA1-DW Social Science

Tuesday, May 27, 2025

14:00 – 17:00

Student Name: ________________________________

Student I.D. #: ________________________________

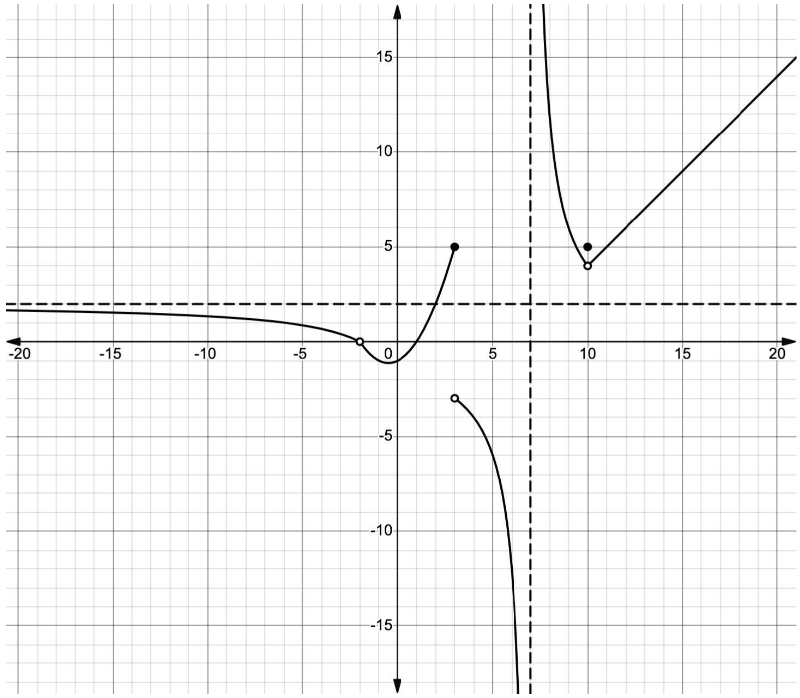
Instructor Name: ________________________________

Instructors: **M. Moodi, A. Jimenez, G. Chu, I. Rajput, G. Bobos-Kristof**

INSTRUCTIONS:

- Print your name, ID number and instructor's name in the space provided above.
- All questions are to be answered in the space provided for each question.
- Give detailed solutions and explain your work clearly.
- Reverse sides of pages may be used for rough work and/or to complete a solution.
- Only calculator permitted is SHARP EL-531***.
- This examination consists of 16 questions page 2 through 12. **Please ensure that you have a complete examination before starting.**

Q-1) Referring to the graph of $f(x)$ below, answer the stated questions:



a) [3 marks] Find the limit at $-2, 3, 7, and\ \ 10$ (Please specify both one-sided limits when the limit does not exist).

Answers:

$\lim_{x\to-2} f(x) = 0$

$\lim_{x\to3} f(x)\ d.n.e.\ \begin{cases} \lim_{x\to3^-} f(x) = 5 \\ \lim_{x\to3^+} f(x) = -3 \end{cases}$

$\lim_{x\to7} f(x)\ d.n.e.\ \begin{cases} \lim_{x\to7^-} f(x) = -\infty \\ \lim_{x\to7^+} f(x) = +\infty \end{cases}$

$\lim_{x\to10} f(x) = 4$

b) [1 mark] Find the limit at $-\infty\ and + \infty$.

$\lim_{x\to-\infty} f(x) = 2$ $\lim_{x\to+\infty} f(x) = +\infty$

c) [4 marks] Find the discontinuities and, for each discontinuity, explain why the function fails to be continuous by referring to the definition of continuity.

Answers:

$x = -2$ $f(-2)\ d.n.e.,$ $x = 3$ $\lim_{x\to3} f(x) = d.n.e.$

$x = 7$ $f(7)\ d.n.e.$ $x = 10$ $\lim_{x\to10} f(x) \neq f(10)$

Q-2) Find the following limits:

a) [5 marks] $\lim_{x\to4} \frac{\sqrt{4x+9}-5}{x-4}$ $\left(\frac{0}{0}\right)$ $Answer$: $\frac{2}{5}$ (≈ 0.4)

b) [5 marks] $\lim_{x\to3} \frac{4x^2-3x-27}{x^3-9x}$ $\left(\frac{0}{0}\right)$ $Answer$: $\frac{7}{6}$ (≈ 1.167)

Q-3) The total box-office receipts for a movie are given by

$$T(x) = 20 + \frac{(4x+1)^2}{4+x^2}$$

where *T*(*x*) is measured in millions of dollars and *x* is the number of months since the movie's release.

a. [2 marks] What are the total box-office receipts after the second month? $Answer: \ T(20) = 30.125$

b. [3 marks] What will the movie gross in the long run?
$Answer: \ \lim_{x\to\infty} T(x) = 36, The\ movie\ will\ gross\ 36\ million\ dollars\ in\ the\ long\ run.$

Q-4) [5 marks] Determine all values of x at which the function *f(x)* is discontinuous and justify your answers.

$$f(x) = \begin{cases} \dfrac{x^2 - x - 6}{x + 2}, x \le 3 \\ 2x + 7, x > 3 \end{cases}$$

$Answer:$ Discontinuous at $x = -2, f(-2) is\ undefined,\ \ dicontinuous\ at\ \ x = 3, \lim_{x\to 3^-} f(x) \ne \lim_{x\to 3^+} f(x)$

Q-5) The gross domestic product (GDP) of a certain country is projected to be

$$N(t) = (t+2)^2, \ \ 0 \le t \le 5$$

billion dollars *t* years from now.

a. [2-marks] Find the average rate of change in GDP between 2 years and 4 years from now.

$$Answer: \frac{N(4) - N(2)}{4 - 2} = 10,$$

$$The\ average\ rate\ of\ change\ in\ GDP\ is\ \$10\ billion\ between\ 2\ and\ 4\ years\ from\ now.$$

b. [4-marks] Use the limit definition of derivative to find the instantaneous rate of change of the GDP 2 years from now.

$$Answer: \ N'(t) = \lim_{h\to 0} \frac{N(t+h) - N(t)}{h} = 2t + 4, \quad N'(2) = 8.$$

$$The\ instantaneous\ rate\ of\ change\ in\ GDP\ is\ 8\ billion\ \$\ per\ year.$$

Q-6) a) [3 marks] Find the derivative of the function $y = \dfrac{tan^2(3x)}{\sqrt{x^2 - 1}}$. DO NOT SIMPLIFY

$Answer:$

$$y' = \frac{\sqrt{x^2-1}\cdot 2\tan(3x)\cdot \sec^2(3x)\cdot 3 - \tan^2(3x)\cdot \frac{1}{2}\left(x^2-1\right)^{-1/2}\cdot 2x}{\left(\sqrt{x^2-1}\right)^2}$$

b) [3 marks] Use logarithmic differentiation to find the derivative of $y = (x^2 + 3)^x$.

Answer:

$$y' = \left[\ln\left(x^2+3\right) + \frac{2x^2}{x^2+3}\right]\left(x^2+3\right)^x$$

Q-7) [5 marks] Given $f(x) = (3x^2 + x)^3$, find $f''(2)$.

Answer: $f''(2) = 17{,}724$

Q-8) [6 marks] The demand function of a certain product is given by

$p = 30\sqrt{106 - x^2}$ $(0 \le x \le 10)$, where p is the unit price in dollars and x is the quantity demanded. Find the marginal revenue when x = 5. $[R(x) = xp]$

Answer: $R'(5) = 186.67\$$

Q-9) [6 marks] The demand equation for a certain brand of USB keys is: $64x^2 + 16p^2 = 10000$ where x represents the number (in thousands) of USB keys demanded each week when the unit price is $\$p$. How fast is the quantity demanded increasing when the unit price per USB key is \$15 and the price is dropping at a rate of \$0.1/USB key/week?

Answer: The demand is increasing at the rate of $37.5 \approx 38$, USB keys per week.

Q-10) During a flu epidemic, the total number of students on a state university campus who contracted influenza by the x^{th} day was given by: $N(x) = \frac{320}{0.2+3e^{-0.01x}}$

a) [2 marks] How many students had influenza initially? Answer: $N(0) = 100$

b) [3 marks] Derive an expression for the rate at which the disease was spreading.

Answer:
$$N'(x) = \frac{9.6e^{-0.01x}}{\left(0.2+3e^{-0.01x}\right)^2}$$

Q-11) [5 marks] Find an equation of the tangent line to the curve $y = \sqrt{x}\ln(2x+1)$ at the point $(1, \ln(3))$.

Answer: $y = 1.216x - 0.117$

Q-12) [5 marks] Find the derivative of the function $f(x) = \frac{sin(\pi x)+cos(\pi x)}{e^{2x}}$, and evaluate its value at $x = 0$.

Answer: $f'(0) = \pi - 2 \quad (\approx 1.14159)$

Q-13) [6 marks] A company's monthly revenue from selling x units of a product is given by:

$$R(x) = -x^2 + 100x$$

The total cost of producing x units is:

$$C(x) = x^2 + 20x + 400 .$$

To maximize profit, how many units should the company sell each month? What is the maximum monthly profit? Answer: $Maximum \ \ when \ \ x = 20, \ \ P(20) = 400\$$

Q-14) Let $f(x) = \frac{4(x-2)}{x^2}$ where the first and the second derivatives are:

$$f'(x) = \frac{-4(x-4)}{x^3} \quad \text{and} \quad f''(x) = \frac{8(x-6)}{x^4}$$

a) [2 mark] Write down the equation(s) of the horizontal and vertical asymptote(s) of $f(x)$, if any.

Answer: $V.A: \ x = 0 \quad ; \quad H.A: \ y = 0$

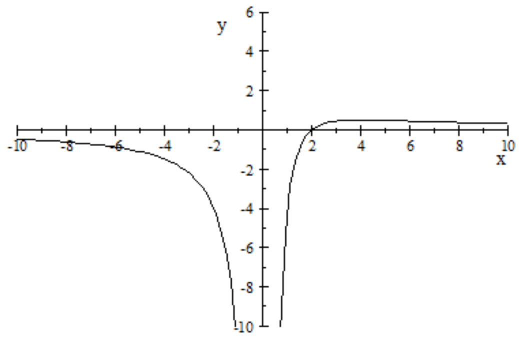
b) [3 marks] Determine the intervals where the function $f(x)$ is increasing/decreasing, and its relative extrema, if any.

Answer: $x \in (-\infty, 0) U (4, +\infty) : f$ decreasing , $x \in (0, 4) : f$ increasing, Relative Maximum : $(4, \frac{1}{2})$

c) [3 marks] Determine the intervals where the function $f(x)$ is concave up/concave down, and its inflection points, if any.

Answer: $x \in (-\infty, 0) U (0, 6) : f$ concave down, $x \in (6, +\infty) : f$ concave up, Inf. point : $(6, \frac{4}{9})$

d) [2 marks] Sketch the graph of $f(x)$.



Q-15) [6 marks] A landscaper plans to create a rectangular garden next to a stream (no fencing needed along the stream). The garden will be divided into two equal parts with interior fencing parallel to the stream. Fencing along the side opposite the stream costs \$10/*ft*, fencing along the two ends perpendicular to the stream cost \$6/*ft*, and the interior dividing fence costs \$8/*ft*. If the total garden area must be exactly 600 square feet, find the dimensions of the garden that minimize the total fencing cost. Answer: *20* ft by *30* ft.

Q-16) [6 marks] The total sales S (in thousands of dollars) of a company is related to the amount of money x (in thousands of dollars) spent on digital advertising by the function:

$$S(x) = -0.001x^3 + 0.045x^2 + 20x + 600 \qquad (0 \leq x \leq 250)$$

Find the inflection point(s) of S(x) and explain its significance in terms of advertising efficiency.

Answer: Inflection point: $(15.0, 906.75)$. The inflection point is at $x = 15.0$, marks the point where sales growth transitions from increasing to diminishing returns, indicating peak advertising efficiency at $15,000$ dollars spent.