Dawson College
Department of Mathematics
Linear Algebra and Vector Geometry for Social Sciences
201-MA3-DW Sections 01 - 05
Final Examination
May 21st, 2026
2:00-5:00 p.m.

Student Name _____

Student I.D. # _____

Teacher _____

Instructors: A. Juhasz, G. Chu, M. Moodi, O. Veres

Instructions:

- Print your name and student ID number in the space provided above.
- This examination contains 14 pages (including this cover page) and 18 questions. Check to see if any pages are missing.
- All questions are to be answered directly on the examination paper in the provided space. If you run out of space for an answer, continue on the back of the page, and please indicate that you have done so.
- Show all your steps and justify your answers.
- Only the **Sharp EL-531**** calculator is permitted.
- This exam booklet must be returned intact.

Question #	Mark
1	/7
2	/5
3	/7
4	/5
5	/3
6	/6
7	/5
8	/4
9	/4
10	/5
11	/3
12	/4
13	/4
14	/4
15	/10
16	/12
17	/5
18	/7
Total	

1. a) *[5+2 marks]* Solve the following system (find the general solution) using Gaussian or Gauss-Jordan Elimination:

$$x_1 + 2x_2 + 4x_3 + 6x_4 = 1$$
$$3x_1 + 7x_2 + 13x_3 + 19x_4 = 4$$
$$-x_1 - 2x_2 + 10x_4 = -5$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 4 & 6 & 1 \\ 3 & 7 & 13 & 19 & 4 \\ -1 & -2 & 0 & 10 & -5 \end{array}\right] \quad r_3 : r_3 - 3r_1,\ r_2 : r_2 + r_1 \rightarrow \quad \left[\begin{array}{cccc|c} 1 & 2 & 4 & 6 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 4 & 16 & -4 \end{array}\right] \quad r_3 : \frac{1}{4} r_3 \rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 4 & 6 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 4 & -1 \end{array}\right] \quad r_1 : r_1 - 4r_3,\ r_2 : r_2 - r_3 \rightarrow \quad \left[\begin{array}{cccc|c} 1 & 2 & 0 & -10 & 5 \\ 0 & 1 & 0 & -3 & 2 \\ 0 & 0 & 1 & 4 & -1 \end{array}\right] \quad r_1 : r_1 - 2r_2 \rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -4 & 1 \\ 0 & 1 & 0 & -3 & 2 \\ 0 & 0 & 1 & 4 & -1 \end{array}\right]$$

$$\begin{cases} x_1 = 1 + 4t \\ x_2 = 2 + 3t \\ x_3 = -1 - 4t \\ x_4 = t \end{cases} \qquad t \in \mathbb{R}$$

b) Find the particular solution when $x_1 = 5$

$1 + 4t = 5$

$4t = 4$

$t = 1$

$(5, 5, -5, 1)$

2. *[5 marks]* Determine the conditions on a such that the system has

a) no solution b) one solution c) infinitely many solutions

$$x + 2y - z = 3$$
$$-3x - 2y + 9z = -8$$
$$8x + 24y + a^2z = a + 24$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ -3 & -2 & 9 & -8 \\ 8 & 24 & a^2 & a+24 \end{array}\right] \quad \begin{array}{l} r_2 : r_2 + 3r_1 \\ r_3 : r_3 - 8r_1 \\ \rightarrow \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 4 & 6 & 1 \\ 0 & 8 & a^2+8 & a \end{array}\right] \quad r_3 : r_3 - 2r_2 \rightarrow \quad \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 4 & 6 & 1 \\ 0 & 0 & a^2-4 & a-2 \end{array}\right]$$

a) no sol'n: $a = -2$

b) one sol'n: $a \in \mathbb{R} \setminus \{-2, 2\}$

c) infinitely many sol'ns: $a = 2$

3. *[4+3 marks]* Let $A = \begin{bmatrix} 1 & 3 & 0 \\ -2 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 3 \end{bmatrix}$ $C = \begin{bmatrix} 0 & -1 & 1 \\ 2 & -2 & 2 \end{bmatrix}$ $D = \begin{bmatrix} 4 & -5 \\ 2 & 1 \end{bmatrix}$

Evaluate (wherever possible):

a) $tr(BAC^T + 3I^7)$

$$BAC^T = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 3 \end{bmatrix}_{2\times 3} \begin{bmatrix} 1 & 3 & 0 \\ -2 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}_{3\times 3} \begin{bmatrix} 0 & 2 \\ -1 & -2 \\ 1 & 2 \end{bmatrix}_{3\times 2}$$

$$= \begin{bmatrix} -1 & 4 & 5 \\ 8 & 3 & 3 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ -1 & -2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ 6 & 16 \end{bmatrix}$$

$$BAC^T + 3I^7 = \begin{bmatrix} 1 & 0 \\ 0 & 16 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 \\ 0 & 19 \end{bmatrix}$$

$$tr(BAC^T + 3I^7) = 4 + 19 = 23$$

b) $p(D)$ where $p(x) = -2x^2 + 3$

$$p(D) = -2D^2 + 3I$$

$$= -2 \begin{bmatrix} 4 & -5 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & -5 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= -2 \begin{bmatrix} 6 & -25 \\ 10 & -9 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -12 & 50 \\ -20 & 18 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -9 & 50 \\ -20 & 21 \end{bmatrix}$$

4. *[5 marks]* Solve for X when: $X\begin{bmatrix} 4 & 2 \\ -1 & 5 \end{bmatrix} = X\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 8 & -2 \\ 1 & 3 \end{bmatrix}$

$$X\left(\begin{bmatrix} 4 & 2 \\ -1 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}\right) = \begin{bmatrix} 8 & -2 \\ 1 & 3 \end{bmatrix}$$

$$X\begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 8 & -2 \\ 1 & 3 \end{bmatrix}$$

$$X = \begin{bmatrix} 8 & -2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 8 & -2 \\ 1 & 3 \end{bmatrix} \cdot \frac{1}{4}\begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 12 & -10 \\ 8 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -5/2 \\ 2 & 1/2 \end{bmatrix}$$

5. [3 marks] If A is an nxn matrix such that $A^3 + 2A + 3I = 0$, show that A is invertible and find A^{-1}.

$$A^3 + 2A = -3I$$

$$A(A^2 + 2I) = -3I$$

$$A\left(-\frac{1}{3}A^2 - \frac{2}{3}I\right) = I$$

$\therefore$ A is invertible

$$A^{-1} = -\frac{1}{3}A^2 - \frac{2}{3}I$$

6. *[6 marks]* Let $A = \begin{bmatrix} 8 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -2 & 7 \end{bmatrix}$ Find A^{-1} using $adj(A)$.

No marks given if this method is not used.

$$adj(A) = \begin{bmatrix} 40 & 15 & 10 \\ 16 & 55 & 18 \\ 8 & 10 & 44 \end{bmatrix}^T$$

$$det(A) = 8(40) - 2(15) - 1(10)$$
$$= 320 - 30 - 10 = 280$$

$$A^{-1} = \frac{1}{det(A)} adj(A) = \frac{1}{280} \begin{bmatrix} 40 & 16 & 8 \\ 15 & 55 & 10 \\ 10 & 18 & 44 \end{bmatrix}$$

7. *[1+4 marks]* Consider the open economy described by the table, where the input is in dollars needed for $1.00 of output.

Input Required per Dollar Output

Provider	Agriculture	Manufacturing	Energy
Agriculture	0.2	0.2	0.1
Manufacturing	0.2	0.4	0.1
Energy	0.1	0.2	0.3

a) Find the consumption matrix for the economy.

$$C = \begin{bmatrix} .2 & .2 & .1 \\ .2 & .4 & .1 \\ .1 & .2 & .3 \end{bmatrix}$$

b) Suppose that the open sector has a demand for $2800 of agriculture, $2520 of manufacturing and $3080 of energy. **Use your solution from Question #6** to find a production vector that will meet this demand. ***No marks given if this method is not used.***

$$I - C = \begin{bmatrix} .8 & -.2 & -.1 \\ -.2 & .6 & -.1 \\ -.1 & -.2 & .7 \end{bmatrix}; \quad d = \begin{bmatrix} 2800 \\ 2520 \\ 3080 \end{bmatrix}$$

$$10(I - C) = \begin{bmatrix} 8 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -2 & 7 \end{bmatrix}$$

$$\vec{x} = [10(I-C)]^{-1} \cdot 10d = \frac{1}{280} \begin{bmatrix} 40 & 16 & 8 \\ 15 & 55 & 10 \\ 10 & 18 & 44 \end{bmatrix} \begin{bmatrix} 28000 \\ 25200 \\ 30800 \end{bmatrix}$$

$$= \begin{bmatrix} 40 & 16 & 8 \\ 15 & 55 & 10 \\ 10 & 18 & 44 \end{bmatrix} \begin{bmatrix} 100 \\ 90 \\ 110 \end{bmatrix} = \begin{bmatrix} 6320 \\ 7550 \\ 7460 \end{bmatrix}$$

8. *[4 marks]* Let A, B, C be invertible matrices. Simplify as much as possible:

$$C(2A^{-1}B^0C^2)^{-1}(6C^TB(A^T)^{-1})^T$$

$= C\left[(C^2)^{-1}A \cdot \frac{1}{2}\right] \cdot 6\left[\left((A^T)^{-1}\right)^T \cdot B^T \cdot C\right]$

$= 3\left(C^{-1}A\right)\left(A^{-1}B^TC\right)$

$= 3C^{-1}B^TC$

9. *[4 marks]* For which values of a does $A = \begin{bmatrix} a & a & 1 \\ 1 & 2 & 1 \\ -1 & 4 & a \end{bmatrix}$ fail to be invertible?

$$\begin{vmatrix} a & a & 1 \\ 1 & 2 & 1 \\ -1 & 4 & a \end{vmatrix} = a(2a-4) - a(a+1) + (6)$$

$= 2a^2 - 4a - a^2 - a + 6$

$= a^2 - 5a + 6$

$= (a-3)(a-2) = 0$

$a = 2$ or $a = 3$

10. *[5 marks]* Let A, B and C be $3x3$ matrices, where $\det(A) = -1$, $\det(B) = 3$ and $\det(C) = -2$. Evaluate $det(M)$ given the following equation:

$$\det(A^T M(2B)^{-1}C^2) = \det(2adj(C) + C^{-1})$$

$$\det(A)\det(M)\frac{1}{\det(2B)}\cdot[\det(C)]^2 = \det\left(2(\det(C))C^{-1} + C^{-1}\right)$$

$$-\det(M)\cdot\frac{1}{2^3\det(B)}\cdot(-2)^2 = \det\left(-4C^{-1} + C^{-1}\right)$$

$$-\det(M)\cdot\frac{1}{2\cdot 3} = \det(-3C^{-1})$$

$$-\frac{1}{6}\det(M) = (-3)^3\det(C^{-1})$$

$$\det M = 27\cdot 6\cdot\frac{1}{\det C} = \frac{27\cdot 6}{-2} = -81$$

11. *[3 marks]* Let $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 3$. Find $\begin{vmatrix} 4g & 4h & 4i \\ d-5a & e-5b & f-5c \\ -a & -b & -c \end{vmatrix}$

$$= \begin{vmatrix} 4g & 4h & 4i \\ d & e & f \\ -a & -b & -c \end{vmatrix} \quad (r_2 : r_2 - 5r_3)$$

$$= -4\begin{vmatrix} g & h & i \\ d & e & f \\ a & b & c \end{vmatrix} = -(-4)\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \quad r_1 \leftrightarrow r_3$$

$$= 4(3) = 12$$

12. *[4 marks]* Let $\vec{u} = (2,-1,1)$ $\vec{v} = (1,1,-2)$ Find all vectors of length 4 perpendicular to both $\vec{u}$ and $\vec{v}$.

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 1 & 1 & -2 \end{vmatrix} = (1, 5, 3)$$

$$\| \vec{u} \times \vec{v} \| = \sqrt{1+25+9} = \sqrt{35}$$

$$w_1 = \frac{4}{\sqrt{35}}(1, 5, 3) = \left(\frac{4}{\sqrt{35}}, \frac{20}{\sqrt{35}}, \frac{12}{\sqrt{35}}\right); \quad w_2 = \frac{4}{\sqrt{35}}(-1, -5, -3)$$

$$= \left(\frac{-4}{\sqrt{35}}, \frac{-20}{\sqrt{35}}, \frac{-12}{\sqrt{35}}\right)$$

13. *[4 marks]* Simplify $(2\vec{u} + \vec{v}) \times (\vec{u} - \vec{v})$.

$$2\vec{u} \times (\vec{u} - \vec{v}) + \vec{v} \times (\vec{u} - \vec{v})$$

$$= 2\vec{u} \times \vec{u} - 2\vec{u} \times \vec{v} + \vec{v} \times \vec{u} - \vec{v} \times \vec{v}$$

($\vec{u} \times \vec{u} = 0$, $\vec{v} \times \vec{v} = 0$)

$$= -2\vec{u} \times \vec{v} - \vec{u} \times \vec{v}$$

$$= -3(\vec{u} \times \vec{v})$$

14. *[4 marks]* Let $\vec{u} = (1,2,4)$ $\vec{v} = (3,2,-1)$ and $\vec{w} = (1,-1,-3)$. Find the volume of the parallelepiped determined by the vectors $3\vec{u}$, $\vec{u} + \vec{v}$, and $2\vec{w}$.

$$3\vec{u} = (3, 6, 12)$$

$$\vec{u} + \vec{v} = (4, 4, 3)$$

$$2\vec{w} = (2, -2, -6)$$

$$V = \text{abs} \begin{vmatrix} 3 & 6 & 12 \\ 4 & 4 & 3 \\ 2 & -2 & -6 \end{vmatrix}$$

$$= |3(-18) - 6(-30) + 12(-16)|$$

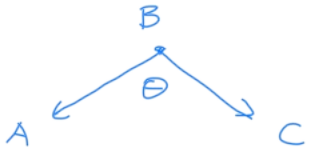
$$= |-54 + 180 - 192| = 66 \text{ u}^3$$

15. *[4+3+3 marks]* Let $A(-1,4,4)$, $B(1,5,3)$, $C(0,2,2)$ be three points in $\mathbb{R}^3$, determine the following:

a) the angle, in degrees, at the vertex B of the triangle ABC (estimated to 2 decimal places).

$\vec{BA} = (-2,-1,1)$ $\quad \|\vec{BA}\| = \sqrt{6}$

$\vec{BC} = (-1,-3,-1)$ $\quad \|\vec{BC}\| = \sqrt{11}$



$$\cos\theta = \frac{(-2,-1,1)\cdot(-1,-3,-1)}{\sqrt{66}} = \frac{2+3-1}{\sqrt{66}}$$

$$\theta = \cos^{-1}\left(\frac{4}{\sqrt{66}}\right) \simeq 60.50^\circ$$

b) the area of the triangle with vertices A, B and C.

$$\vec{BA} \times \vec{BC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & -1 & 1 \\ -1 & -3 & -1 \end{vmatrix} = (4,-3,5)$$

$$A = \frac{\sqrt{16+9+25}}{2} = \frac{\sqrt{50}}{2} = \frac{5\sqrt{2}}{2} \quad u^2$$

c) the general equation of the plane containing the three points.

$\vec{n} = (4,-3,5)$ $\quad$ using $C(0,2,2)$:

$$4x - 3(y-2) + 5(z-2) = 0$$

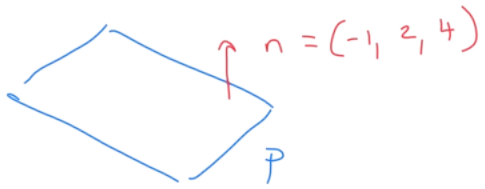
$$4x - 3y + 6 + 5z - 10 = 0$$

$$4x - 3y + 5z - 4 = 0$$

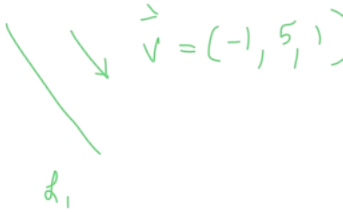
16. *[4 marks each]* Given the plane $\wp_1: -x + 2y + 4z = 9$ and two lines $\mathcal{L}_1: \begin{cases} x = 4 - t \\ y = 1 + 5t \\ z = -1 + t \end{cases} t\epsilon\mathbb{R}$

and $\mathcal{L}_2: \begin{cases} x = 2 - 3s \\ y = -9 + 5s \\ z = -3 + s \end{cases} s\epsilon\mathbb{R}$

a) Determine whether $\mathcal{L}_1$ and $\wp_1$ are perpendicular.



$\vec{v} \neq k\vec{n}$ ∴ L_1 & P_1 are not perpendicular



b) Find the point of intersection between the line $\mathcal{L}_1$ and the plane $\wp_1$.

$-(4-t) + 2(1+5t) + 4(-1+t) = 9$

$-4 + t + 2 + 10t - 4 + 4t = 9$

$15t - 6 = 9$

$15t = 15 \Rightarrow t = 1$ → pt of intersection

(plugging into l_1) $= (3, 6, 0)$

c) Do $\mathcal{L}_1$ and $\mathcal{L}_2$ intersect? If yes, find the point of intersection between $\mathcal{L}_1$ and $\mathcal{L}_2$.

$2 - 3s = 4 - t \Rightarrow -3s + t = 2$

$-9 + 5s = 1 + 5t \Rightarrow 5s - 5t = 10 \Rightarrow s - t = 2$

$-3 + s = -1 + t \Rightarrow s - t = 2$ ← same

$-3s + t = 2$

$s - t = 2$

$-2s = 4 \Rightarrow s = -2$

$t = -4$

plugging s into l_2: $(8, -19, -5)$

(or t into l_1: $(8, -19, -5)$)

17. *[1+4 marks]* Consider the transition matrix $P = \begin{bmatrix} .8 & .25 \\ .2 & .75 \end{bmatrix}$. State why P is regular and find its steady state vector.

P is regular since all values are positive.

$$[I-P \mid 0] = \left[\begin{array}{cc|c} .2 & -.25 & 0 \\ -.2 & .25 & 0 \end{array}\right] \quad r_2 : r_2 + r_1 \rightarrow$$

$$\left[\begin{array}{cc|c} .2 & -.25 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad r_1 : r_1 \div 0.2 \rightarrow$$

$$\left[\begin{array}{cc|c} 1 & -5/4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\vec{q} = \begin{bmatrix} 5/4 \\ 1 \end{bmatrix} t$$

$$\frac{5}{4}t + t = 1$$

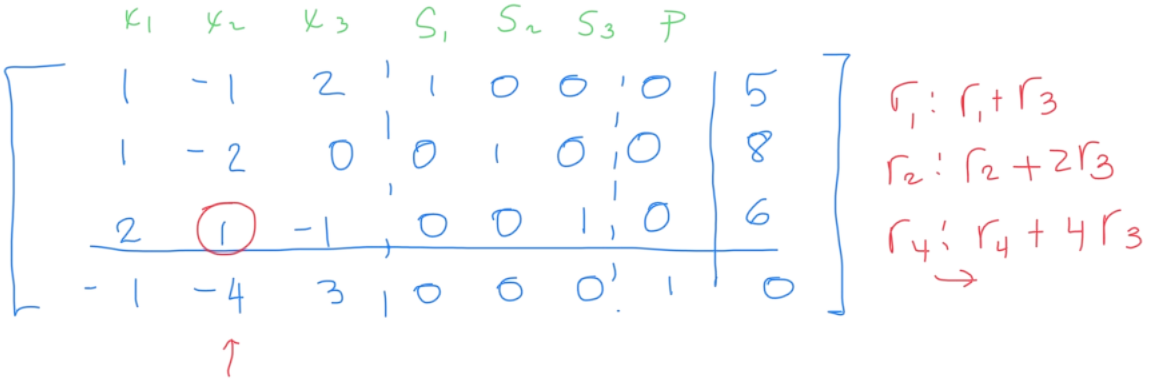
$$\frac{9}{4}t = 1 \Rightarrow t = \frac{4}{9}$$

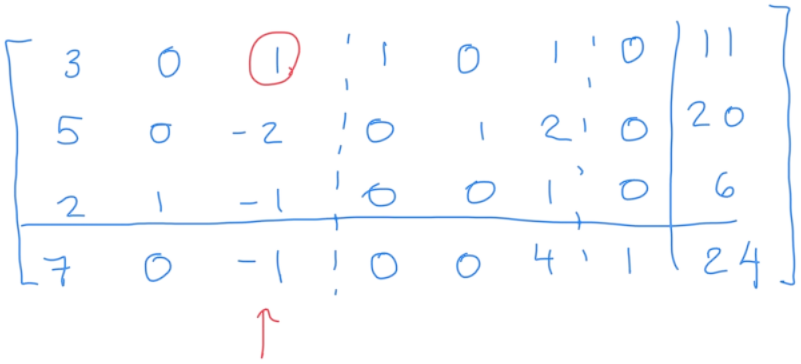
$$\vec{q} = \begin{bmatrix} 5/9 \\ 4/9 \end{bmatrix}$$

18. *[7 marks]* Maximize $P = x_1 + 4x_2 - 3x_3$ subject to the constraints:

$$\begin{cases} x_1 - x_2 + 2x_3 \leq 5 \\ x_1 - 2x_2 \leq 8 \\ 2x_1 + x_2 - x_3 \leq 6 \end{cases}$$

$$x_1, x_2, x_3 \geq 0$$







$x_1 = 0$
$x_2 = 17$
$x_3 = 11$
$P = 35$