Dawson College

Mathematics Department

FINAL EXAMINATION FOR WINTER 2016

CALCULUS I (Electronics Engineering Technology)

201-NYA-05 section 4

May 17, 2016 2:00pm - 5:00pm

Student Name: (Solutions)

Student I.D. #: ____________________

Instructor: Alexander Hariton

Time: 3 hours

Instructions:

- Print your name and student ID number in the space provided above.
- All questions are to be answered directly on the examination paper in the space provided. Show your complete work and give explanations.
- A Sharp EL-531XG calculator is permitted.

This examination consists of 11 questions. Please ensure that you have a complete examination.

This examination must be returned intact.

1. **(16 marks)**

Evaluate the following limits. Show your work.

(a) $\lim_{x\to 5} \dfrac{x^2+2x-35}{x^2-13x+40}$ $= \lim_{x\to 5} \dfrac{(x+7)\cancel{(x-5)}}{(x-8)\cancel{(x-5)}}$

$= \lim_{x\to 5} \dfrac{x+7}{x-8} = \dfrac{5+7}{5-8} = \dfrac{12}{-3} = \boxed{-4}$

(b) $\lim_{x\to 3} \dfrac{\sqrt{3x-5}-2}{3-x}$ $= \lim_{x\to 3} \dfrac{\sqrt{3x-5}-2}{3-x} \cdot \dfrac{\sqrt{3x-5}+2}{\sqrt{3x-5}+2}$

$= \lim_{x\to 3} \dfrac{(3x-5)-4}{(3-x)(\sqrt{3x-5}+2)} = \lim_{x\to 3} \dfrac{3x-9}{(3-x)(\sqrt{3x-5}+2)}$

$= \lim_{x\to 3} \dfrac{3\cancel{(x-3)}}{-\cancel{(x-3)}(\sqrt{3x-5}+2)} = \lim_{x\to 3} \dfrac{-3}{\sqrt{3x-5}+2}$

$= \dfrac{-3}{\sqrt{3(3)-5}+2} = \dfrac{-3}{\sqrt{4}+2} = \boxed{-\dfrac{3}{4}}$

1. (continued)

(c) $\lim_{x\to 6^-} \frac{3x^2-18x}{|x-6|}$

If $x \to 6^-$, $x < 6$, so $x-6<0$

so $|x-6| = -(x-6)$

$$\lim_{x\to 6^-} \frac{3x^2-18x}{|x-6|} = \lim_{x\to 6^-} \frac{3x\cancel{(x-6)}}{-\cancel{(x-6)}} = \lim_{x\to 6^-} (-3x)$$

$$= -3(6) = \boxed{-18}$$

(d) $\lim_{x\to -\infty} \frac{2x^3-x^2+8}{x(5-x^2)}$ $= \lim_{x\to -\infty} \frac{2x^3-x^2+8}{5x-x^3} \cdot \frac{\left(\frac{1}{x^3}\right)}{\left(\frac{1}{x^3}\right)}$

$$= \lim_{x\to -\infty} \frac{2-\frac{1}{x}+\frac{8}{x^3}}{\frac{5}{x^2}-1} = \frac{2-0+0}{0-1} = \boxed{-2}$$

2. (16 marks)

Find the derivatives of the following functions. Simplify your answers.

(a) $f(x) = e^{4x-1} \cos(x^4 + 5)$

$$f'(x) = \frac{d}{dx}\left(e^{4x-1}\right) \cdot \cos(x^4+5) + e^{4x-1} \cdot \frac{d}{dx}\left[\cos(x^4+5)\right]$$

$$= e^{4x-1} \cdot \frac{d}{dx}(4x-1) \cdot \cos(x^4+5) + e^{4x-1} \cdot \left[-\sin(x^4+5)\right] \cdot \frac{d}{dx}(x^4+5)$$

$$= e^{4x-1} \cdot (4) \cdot \cos(x^4+5) - e^{4x-1} \cdot \sin(x^4+5) \cdot (4x^3)$$

$$= \boxed{4e^{4x-1}\left[\cos(x^4+5) - x^3 \sin(x^4+5)\right]}$$

(b) $f(x) = \ln\left(\sin\left(e^{\sqrt{x}}\right)\right)$

$$f'(x) = \frac{1}{\sin(e^{\sqrt{x}})} \cdot \frac{d}{dx}\left[\sin(e^{\sqrt{x}})\right]$$

$$= \frac{1}{\sin(e^{\sqrt{x}})} \cdot \cos(e^{\sqrt{x}}) \cdot \frac{d}{dx}\left[e^{\sqrt{x}}\right]$$

$$= \frac{\cos(e^{\sqrt{x}})}{\sin(e^{\sqrt{x}})} \cdot e^{\sqrt{x}} \cdot \frac{d}{dx}(\sqrt{x})$$

$$= \cot(e^{\sqrt{x}}) \cdot e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} = \boxed{\frac{e^{\sqrt{x}} \cot(e^{\sqrt{x}})}{2\sqrt{x}}}$$

2. (continued)

(c) $f(x) = \sec^3 x \tan^5 x$

$$f'(x) = \frac{d}{dx}\left[\sec^3 x\right] \cdot \tan^5 x + \sec^3 x \cdot \frac{d}{dx}\left[\tan^5 x\right]$$

$$= 3\sec^2 x \cdot \frac{d}{dx}(\sec x) \cdot \tan^5 x + \sec^3 x \cdot 5\tan^4 x \cdot \frac{d}{dx}(\tan x)$$

$$= 3\sec^2 x \sec x \tan x \tan^5 x + \sec^3 x \,(5) \tan^4 x \sec^2 x$$

$$= 3\sec^3 x \tan^6 x + 5\sec^5 x \tan^4 x$$

$$= \boxed{\sec^3 x \tan^4 x \,(3\tan^2 x + 5\sec^2 x)}$$

(d) $f(x) = 2\arctan(3x) - \ln(9x^2+1) + 8$

$$f'(x) = 2\frac{d}{dx}\left[\arctan(3x)\right] - \frac{1}{9x^2+1} \cdot \frac{d}{dx}(9x^2+1) + 0$$

$$= 2\left(\frac{1}{1+(3x)^2}\right) \cdot \frac{d}{dx}(3x) - \frac{1}{9x^2+1}(18x)$$

$$= \frac{2}{1+9x^2}(3) - \frac{18x}{9x^2+1} = \frac{6-18x}{9x^2+1}$$

$$= \boxed{\frac{6(1-3x)}{9x^2+1}}$$

3. (4 marks)

Use only the limit definition of the derivative to find $f'(x)$ if

$$f(x) = \sqrt{3x+2}$$

Important: No marks will be given if you do not use the limit definition of the derivative.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{3(x+h)+2} - \sqrt{3x+2}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{3x+3h+2} - \sqrt{3x+2}}{h} \cdot \frac{\sqrt{3x+3h+2} + \sqrt{3x+2}}{\sqrt{3x+3h+2} + \sqrt{3x+2}}$$

$$= \lim_{h \to 0} \frac{(\cancel{3x} + 3h + \cancel{2}) - (\cancel{3x} + \cancel{2})}{h\left(\sqrt{3x+3h+2} + \sqrt{3x+2}\right)}$$

$$= \lim_{h \to 0} \frac{3\cancel{h}}{\cancel{h}\left(\sqrt{3x+3h+2} + \sqrt{3x+2}\right)}$$

$$= \lim_{h \to 0} \frac{3}{\sqrt{3x+3h+2} + \sqrt{3x+2}}$$

$$= \frac{3}{\sqrt{3x+0+2} + \sqrt{3x+2}} = \boxed{\frac{3}{2\sqrt{3x+2}}}$$

4. (8 marks)

(a) Find the equation of the line tangent to the curve $f(x) = \frac{x}{x-3}$ at the point $(4,4)$

$$f'(x) = \frac{\frac{d}{dx}(x)\cdot(x-3) - x\cdot\frac{d}{dx}(x-3)}{(x-3)^2} = \frac{x-3-x}{(x-3)^2} = \frac{-3}{(x-3)^2}$$

The slope of the tangent line at point $(4,4)$ (where $x=4$) is:

$$m_t = f'(4) = \frac{-3}{(4-3)^2} = \frac{-3}{1} = -3$$

So, using point $(4,4)$, we find the equation of the tangent line: $y-4 = -3(x-4)$, so $y-4 = -3x+12$

$$\boxed{y = -3x+16}$$

(b) Find the equation of the line normal to the curve $f(x) = \frac{3}{\sqrt{x^2+5}}$ at the point $(2,1)$

$$f'(x) = \frac{\frac{d}{dx}(3)\cdot\sqrt{x^2+5} - 3\cdot\frac{d}{dx}(\sqrt{x^2+5})}{(\sqrt{x^2+5})^2} = \frac{0 - \frac{3}{2\sqrt{x^2+5}}(2x)}{x^2+5}$$

$$= \boxed{-\frac{3x}{(x^2+5)^{3/2}}}$$, so the slope of the tangent

at $(2,1)$ is : $m_t = -\frac{3(2)}{(2^2+5)^{3/2}} = -\frac{6}{9^{3/2}} = \frac{-6}{27} = -\frac{2}{9}$

so the slope of the normal line is $\boxed{m_n = \frac{9}{2}}$

The equation of the normal line is: $y-1 = \frac{9}{2}(x-2)$, so

$y-1 = \frac{9}{2}x - 9$. $\boxed{y = \frac{9}{2}x - 8}$ or $\boxed{9x-2y-16=0}$

5. (8 marks)

(a) Find $\frac{dy}{dx}$ for the equation $x^2y^3 - 2x^3 = x\cos y + 6$

$$\frac{d}{dx}(x^2)\cdot y^3 + x^2\cdot\frac{d}{dx}(y^3) - 2\cdot\frac{d}{dx}(x^3) = \frac{d}{dx}(x)\cdot\cos y + x\cdot\frac{d}{dx}(\cos y) + 0$$

$$2xy^3 + 3x^2y^2\frac{dy}{dx} - 6x^2 = \cos y - x\sin y\frac{dy}{dx}$$

$$3x^2y^2\frac{dy}{dx} + x\sin y\frac{dy}{dx} = \cos y - 2xy^3 + 6x^2$$

$$(3x^2y^2 + x\sin y)\frac{dy}{dx} = \cos y - 2xy^3 + 6x^2$$

$$\frac{dy}{dx} = \frac{\cos y - 2xy^3 + 6x^2}{3x^2y^2 + x\sin y} = \boxed{\frac{\cos y - 2xy^3 + 6x^2}{x(3xy^2 + \sin y)}}$$

(b) Find $\frac{d^2y}{dx^2}$ for the equation $4x^4 + y^4 = 2$

$$4\cdot\frac{d}{dx}(x^4) + \frac{d}{dx}(y^4) = 0, \text{ so } 16x^3 + 4y^3\frac{dy}{dx} = 0,$$

$$4y^3\frac{dy}{dx} = -16x^3, \text{ so } \frac{dy}{dx} = -\frac{16x^3}{4y^3}, \quad \boxed{\frac{dy}{dx} = \frac{-4x^3}{y^3}}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(-\frac{4x^3}{y^3}\right) = -4\frac{d}{dx}\left(\frac{x^3}{y^3}\right) = -4\left[\frac{\frac{d}{dx}(x^3)\cdot y^3 - x^3\cdot\frac{d}{dx}(y^3)}{(y^3)^2}\right]$$

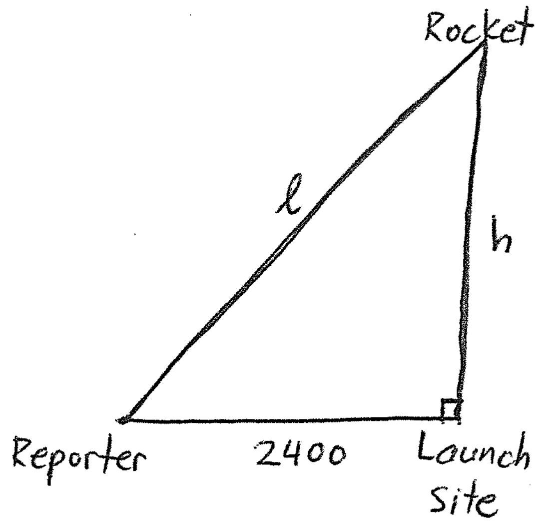
$$= -\frac{4}{y^6}\left[3x^2y^3 - x^3(3y^2)\frac{dy}{dx}\right] = -\frac{4}{y^6}\left[3x^2y^3 - 3x^3y^2\left(-\frac{4x^3}{y^3}\right)\right]$$

$$= -\frac{4}{y^6}\left[\frac{3x^2y^4}{y} + \frac{12x^6}{y}\right] = -\frac{4}{y^6}\left(\frac{3x^2}{y}\right)\left[y^4 + 4x^4\right]$$

$$= -\frac{12x^2}{y^7}(2) = \boxed{-\frac{24x^2}{y^7}}$$

6. (6 marks)

A newspaper reporter stands at a point on the ground 2400 m from the site where a rocket is being launched. At a certain instant, the rocket begins its ascent, and rises vertically at a constant speed of 200 m/s. At a time 16 seconds later, how fast is the distance between the reporter and the rocket increasing?



Let h be the height of the rocket above the ground (in m) and l be the distance between the reporter and the rocket (in m).

$\frac{dh}{dt} = 200$ m/s

$\frac{dl}{dt}$ is what we must determine.

Pythagorean Theorem: $\boxed{l^2 = h^2 + (2400)^2}$

so $2l \frac{dl}{dt} = 2h \frac{dh}{dt} + 0$, $\boxed{\frac{dl}{dt} = \frac{h}{l}\frac{dh}{dt}}$

After 16 seconds, $h = (200 \text{ m/s})(16 \text{ s}) = \boxed{3200 \text{ m}}$

$l^2 = h^2 + (2400)^2 = (3200)^2 + (2400)^2 = 16\,000\,000$

so $\boxed{l = 4000 \text{ m}}$

$$\frac{dl}{dt} = \frac{h}{l}\frac{dh}{dt} = \frac{3200}{4000}(200) = \boxed{160}$$

The distance between the reporter and the rocket is increasing at the rate of $\boxed{160 \text{ m/s}}$

7. (14 marks)

For the function $f(x) = \dfrac{x^2}{(x-2)^2}$ find:

(i) the x and y intercepts (if any)

(ii) the horizontal and vertical asymptotes (if any)

(iii) the intervals where $f(x)$ is increasing and where $f(x)$ is decreasing

(iv) the relative maxima and minima (if any)

(v) the intervals where $f(x)$ is concave up and where $f(x)$ is concave down

(vi) the points of inflection (if any)

Use the above information to sketch the graph of the function $f(x)$

Note: $f'(x) = \dfrac{-4x}{(x-2)^3}$ $\quad f''(x) = \dfrac{8(x+1)}{(x-2)^4}$

(i) $f(0) = \dfrac{0^2}{(0-2)^2} = \dfrac{0}{4} = 0$, so $\boxed{(0,0)}$ is the y-intercept

$f(x) = 0$ if $\dfrac{x^2}{(x-2)^2} = 0$, $\underline{x=0}$, so $\boxed{(0,0)}$ is the only x-intercept.

(ii) Horizontal asymptotes: $f(x) = \dfrac{x^2}{x^2-4x+4} = \dfrac{1}{1-\frac{4}{x}+\frac{4}{x^2}}$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1}{1-\frac{4}{x}+\frac{4}{x^2}} = \frac{1}{1-0+0} = 1$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{1}{1-\frac{4}{x}+\frac{4}{x^2}} = \frac{1}{1-0+0} = 1$$

So $\boxed{y=1}$ is a horizontal asymptote.

Vertical asymptotes: $f(x) = \dfrac{x^2}{(x-2)^2}$ undefined at $x=2$

$\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-} \dfrac{x^2}{(x-2)^2} \to \dfrac{(2)^2}{0^+} \to \infty$, so $\boxed{\lim\limits_{x\to 2^-} f(x) = \infty}$

$\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} \dfrac{x^2}{(x-2)^2} \to \dfrac{(2)^2}{0^+} \to \infty$, so $\boxed{\lim\limits_{x\to 2^+} f(x) = \infty}$

so $\boxed{x=2}$ is a vertical asymptote

7. (continued)

(iii) $f'(x) = \dfrac{-4x}{(x-2)^3}$, so $f'(x) = 0$ if $x=0$ and $f'(x)$ does not exist if $x=2$

Interval	$-4x$	$(x-2)^3$	$f'(x)$	f
$x<0$	$+$	$-$	$-$	$\searrow$
$0<x<2$	$-$	$-$	$+$	$\nearrow$
$x>2$	$-$	$+$	$-$	$\searrow$

$f(x)$ is increasing on $(0,2)$ and decreasing on $(-\infty,0)$ and $(2,\infty)$.

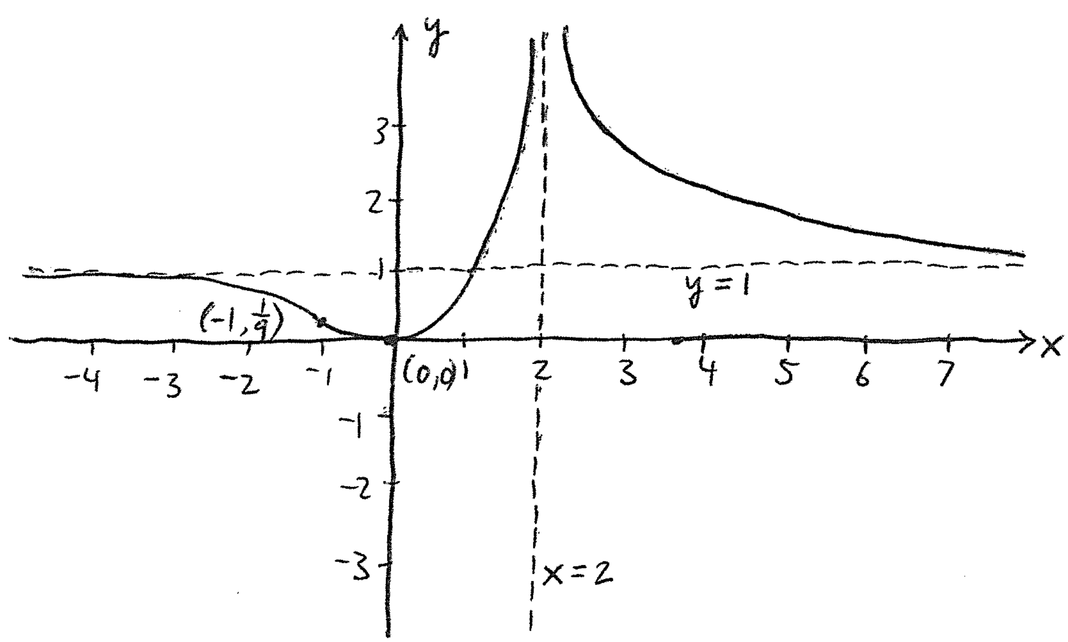
(iv) Relative minimum at $x=0$: $f(0)=0$, so at $\boxed{(0,0)}$
At $x=2$, there is no relative maximum or minimum since it is a vertical asymptote.

(v) $f''(x) = \dfrac{8(x+1)}{(x-2)^4}$, so $f''(x)=0$ if $x=-1$ and $f''(x)$ does not exist if $x=2$

Interval	8	$x+1$	$(x-2)^4$	$f''(x)$	f
$x<-1$	$+$	$-$	$+$	$-$	$\cap$
$-1<x<2$	$+$	$+$	$+$	$+$	$\cup$
$x>2$	$+$	$+$	$+$	$+$	$\cup$

$f(x)$ is concave up on $(-1,2)$ and $(2,\infty)$ and concave down on $(-\infty,-1)$.

(vi) Point of inflection at $x=-1$: $f(-1) = \dfrac{(-1)^2}{(-1-2)^2} = \dfrac{1}{9}$: at $\boxed{\left(-1,\frac{1}{9}\right)}$



8. (6 marks)

An electric circuit contains a resistor whose resistance R varies with time t (measured in s) according to the equation $R(t) = \dfrac{t}{(t+1)^2}$ (in Ω).

If the electric current through the circuit is $i(t) = \dfrac{t+1}{\sqrt{t^2+4}}$ (in A), at what time t is the power $P = i^2 R$ in the resistor a maximum?

Hint: simplify the expression for $P(t)$ before attempting to take the derivative.

The power in the resistor is :

$$P = i^2 R = \left(\frac{t+1}{\sqrt{t^2+4}}\right)^2 \left(\frac{t}{(t+1)^2}\right) = \frac{\cancel{(t+1)^2}}{t^2+4} \cdot \frac{t}{\cancel{(t+1)^2}}$$

$$= \frac{t}{t^2+4}$$

$\boxed{P(t) = \dfrac{t}{t^2+4}}$ is the function to be maximized.

$$P'(t) = \frac{(1)(t^2+4) - t(2t)}{(t^2+4)^2} = \frac{t^2+4-2t^2}{(t^2+4)^2} = \boxed{\frac{4-t^2}{(t^2+4)^2}}$$

so $P'(t) = 0$ if $4 - t^2 = 0$, $t^2 = 4$, $\boxed{t = 2}$

(since t is positive)

If $\underline{0 < t < 2}$, $t^2 < 4$, so $0 < 4 - t^2$, $4 - t^2 > 0$, $\dfrac{4-t^2}{(t^2+4)^2} > 0$, $\underline{P'(t) > 0}$

If $\underline{t > 2}$, $t^2 > 4$, so $0 > 4 - t^2$, $4 - t^2 < 0$, $\dfrac{4-t^2}{(t^2+4)^2} < 0$, $\underline{P'(t) < 0}$

so $P(t)$ has its maximum at $\boxed{t = 2}$

The power in the resistor is a maximum when $\boxed{t = 2 \text{ seconds}}$

9. (8 marks)

Evaluate the following indefinite integrals

(a) $\displaystyle\int \left(12x^3 - 6\sec^2 x + \frac{5}{\sqrt{1-x^2}}\right) dx$

$$= 12\int x^3 dx - 6\int \sec^2 x \, dx + 5\int \frac{1}{\sqrt{1-x^2}} dx$$

$$= 12\left(\frac{x^4}{4}\right) - 6(\tan x) + 5(\arcsin x) + C$$

$$= \boxed{3x^4 - 6\tan x + 5\arcsin x + C}$$

(b) $\displaystyle\int \frac{x^4 - 2x}{(x^5 - 5x^2 + 7)^3} dx$

Let $u = x^5 - 5x^2 + 7$, so $du = (5x^4 - 10x)\,dx$

$$\frac{1}{5} du = (x^4 - 2x)\,dx$$

$$\int \frac{x^4 - 2x}{(x^5 - 5x^2 + 7)^3} dx = \int \frac{1}{u^3}\left(\frac{1}{5} du\right) = \frac{1}{5}\int u^{-3} du$$

$$= \frac{1}{5} \frac{u^{-2}}{-2} + C = -\frac{1}{10u^2} + C$$

$$= \boxed{-\frac{1}{10(x^5 - 5x^2 + 7)^2} + C}$$

10. (8 marks)

Evaluate the following definite integrals

(a) $\displaystyle\int_1^3 \left(5 + \frac{2}{x} - \frac{1}{x^2}\right) dx$

$$\int \left(5 + \frac{2}{x} - \frac{1}{x^2}\right) dx = \int \left(5 + \frac{2}{x} - x^{-2}\right) dx$$

$$= 5x + 2\ln|x| - \frac{x^{-1}}{-1} = \boxed{5x + 2\ln|x| + \frac{1}{x}}$$

so $\displaystyle\int_1^3 \left(5 + \frac{2}{x} - \frac{1}{x^2}\right) dx = \left(5x + 2\ln|x| + \frac{1}{x}\right)\Big|_1^3$

$$= \left[5(3) + 2\ln(3) + \frac{1}{3}\right] - \left[5(1) + 2\ln(1) + \frac{1}{1}\right]$$

$$= 15 + 2\ln 3 + \frac{1}{3} - 5 - 0 - 1 = 9 + \frac{1}{3} + 2\ln 3$$

$$= \boxed{\frac{28}{3} + 2\ln 3} \approx \boxed{11.53}$$

(b) $\displaystyle\int_0^1 (2x^3+1)\sqrt{x^4+2x+1}\, dx$

Let $u = x^4 + 2x + 1$, so $du = (4x^3+2)\,dx$

$$\frac{1}{2} du = (2x^3+1)\,dx$$

$$\int (2x^3+1)\sqrt{x^4+2x+1}\,dx = \int \sqrt{u}\left(\frac{1}{2}du\right) = \frac{1}{2}\int u^{1/2}\,du$$

$$= \frac{1}{2}\frac{u^{3/2}}{\left(\frac{3}{2}\right)} = \frac{1}{2}\left(\frac{2}{3}\right)u^{3/2} = \frac{1}{3}u^{3/2} = \boxed{\frac{1}{3}(x^4+2x+1)^{3/2}}$$

so $\displaystyle\int_0^1 (2x^3+1)\sqrt{x^4+2x+1}\,dx = \frac{1}{3}(x^4+2x+1)^{3/2}\Big|_0^1$

$$= \frac{1}{3}(1+2+1)^{3/2} - \frac{1}{3}(0+0+1)^{3/2} = \frac{1}{3}(4)^{3/2} - \frac{1}{3}(1)^{3/2}$$

$$= \frac{1}{3}(8) - \frac{1}{3}(1) = \boxed{\frac{7}{3}}$$

11. (6 marks)

A 120-μF capacitor initially has a voltage of 100 V across it. At a certain instant (when t=0), a current $i(t) = 60t$ (where i is measured in mA and t is measured in s) is sent through the circuit containing the capacitor. How long does it take for the voltage across the capacitor to reach 150 V ?

$$V_c = \frac{1}{C}\int i\,dt = \frac{1}{120\times 10^{-6}}\int (60t)\times 10^{-3}\,dt$$

$$= 500\int t\,dt = 500\left(\frac{t^2}{2}\right) + C_1 = \boxed{250t^2 + C_1}$$

At $t=0$, $V_c = 100$, so $100 = 250(0)^2 + C_1 = C_1$

so $\boxed{C_1 = 100}$

$$\boxed{V_c = 250t^2 + 100}$$

We must find t for which $V_c = 150$:

$$150 = 250t^2 + 100$$

$$250t^2 = 50$$

$$t^2 = 0.2, \text{ so } \boxed{t = 0.45}$$

The voltage across the capacitor will reach 150 V in $\boxed{0.45 \text{ seconds}}$