Dawson College

Mathematics Department

FINAL EXAMINATION FOR FALL 2023 (Version 1)

CALCULUS I (Science)

201-NYA-05

Sections: 1, 3, 4, 8, 9, 10, 11, 12

January 8, 2024 2:00pm – 5:00pm

Student Name: (Solutions)

Student ID #: ______________________

Instructors: R. Acteson, K. Adl-Zarabi, A. Gambioli, A. Jimenez

Time: 3 hours

Instructions:

Print your name and student ID number in the space provided above.

All questions are to be answered directly on the examination paper in the space provided. Show your complete work and give explanations.

A Sharp EL-531** calculator is permitted.

This examination consists of 14 questions. Please ensure that you have a complete examination.

This examination must be returned intact.

1. (12 marks)

Evaluate each of the following limits. Do not use L'Hopital's Rule.

(a) $\lim_{x \to -2} \frac{3x^2 + 7x + 2}{x^2 - 5x - 14}$

$$= \lim_{x \to -2} \frac{(3x+1)\cancel{(x+2)}}{(x-7)\cancel{(x+2)}} = \lim_{x \to -2} \frac{3x+1}{x-7}$$

$$= \frac{3(-2)+1}{-2-7} = \frac{-5}{-9} = \boxed{\frac{5}{9}}$$

(b) $\lim_{x \to 0} \frac{\sin(5x)\ \tan(7x)}{x^2}$

$$= \lim_{x \to 0} \left(\frac{\sin(5x)}{x} \cdot \frac{5}{5} \quad \frac{\tan(7x)}{x} \cdot \frac{7}{7} \right)$$

$$= \left(\lim_{x \to 0} 5 \ \frac{\sin(5x)}{5x} \right) \left(\lim_{x \to 0} 7 \ \frac{\tan(7x)}{7x} \right)$$

$$= 5(1)\,7(1) = \boxed{35}$$

1. (continued)

(c) $\lim_{x \to 3} \dfrac{\sqrt{x+1} - 2}{\sqrt{2x-5} - 1}$

$$= \lim_{x \to 3} \frac{\sqrt{x+1} - 2}{\sqrt{2x-5} - 1} \cdot \frac{\sqrt{x+1} + 2}{\sqrt{x+1} + 2} \cdot \frac{\sqrt{2x-5} + 1}{\sqrt{2x-5} + 1}$$

$$= \lim_{x \to 3} \frac{(x+1-4)(\sqrt{2x-5} + 1)}{(2x-5-1)(\sqrt{x+1} + 2)}$$

$$= \lim_{x \to 3} \frac{\cancel{(x-3)}(\sqrt{2x-5} + 1)}{2\cancel{(x-3)}(\sqrt{x+1} + 2)}$$

$$= \lim_{x \to 3} \frac{\sqrt{2x-5} + 1}{2(\sqrt{x+1} + 2)} = \frac{\sqrt{2(3)-5} + 1}{2(\sqrt{3+1} + 2)} = \frac{\sqrt{1} + 1}{2(2+2)}$$

$$= \frac{2}{2(4)} = \boxed{\frac{1}{4}}$$

2. (8 marks)

Evaluate each of the following limits using L'Hopital's Rule

(a) $\lim_{x\to 0^+} (1+\sin(3x))^{1/x}$

Let $y = (1+\sin(3x))^{\frac{1}{x}}$, so $\ln y = \frac{1}{x}\ln(1+\sin(3x))$

$$\lim_{x\to 0^+} (\ln y) = \lim_{x\to 0^+} \frac{\ln(1+\sin(3x))}{x} \to \frac{0}{0}$$

$$\text{L'HR: } = \lim_{x\to 0^+} \frac{\frac{1}{1+\sin(3x)}\cdot\cos(3x)\cdot 3}{(1)} = \lim_{x\to 0^+} \frac{3\cos(3x)}{1+\sin(3x)}$$

$$= \frac{3\cos(0)}{1+\sin(0)} = \frac{3(1)}{1+0} = 3$$

$$\text{so } \lim_{x\to 0^+} (1+\sin(3x))^{\frac{1}{x}} = \boxed{e^3}$$

(b) $\lim_{x\to 1^+} \left(\frac{1}{\ln x} - \frac{x}{x-1}\right)$

$$= \lim_{x\to 1^+} \left(\frac{x-1-x\ln x}{(x-1)\ln x}\right) \to \frac{1-1-\ln 1}{(1-1)\ln 1} \to \frac{0}{0}$$

$$\text{L'HR: } = \lim_{x\to 1^+} \frac{1-\ln x - x\left(\frac{1}{x}\right)}{\ln x + \frac{x-1}{x}} = \lim_{x\to 1^+} \frac{-\ln x}{\ln x + 1 - \frac{1}{x}} \to \frac{0}{0}$$

$$\text{L'HR: } = \lim_{x\to 1^+} \frac{-\frac{1}{x}}{\frac{1}{x}+\frac{1}{x^2}} = \frac{-1}{1+1} = \boxed{-\frac{1}{2}}$$

3. (6 marks)

Find the values of a and b that make the function

$$f(x) = \begin{cases} \frac{x^2-1}{x+1} & \text{if} \quad x < -1 \\ ax + b & \text{if} \quad -1 \le x \le 2 \\ ax^2 - 3bx + 1 & \text{if} \quad x > 2 \end{cases}$$

continuous on $(-\infty, \infty)$

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} \frac{x^2-1}{x+1} = \lim_{x\to -1^-} \frac{\cancel{(x+1)}(x-1)}{\cancel{x+1}} = \lim_{x\to -1^-} (x-1)$$

$$= -1-1 = -2$$

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} (ax+b) = -a+b$$

so $-a+b = -2$, so $\underline{b = a-2}$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (ax+b) = 2a+b$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (ax^2-3bx+1) = 4a-6b+1$$

so $4a-6b+1 = 2a+b$

$$2a - 7b + 1 = 0$$

$$2a - 7(a-2) + 1 = 0$$

$-5a + 15 = 0$, so $\boxed{a=3}$

and $b = 3-2$, so $\boxed{b=1}$

4. (4 marks)

Use the limit definition of the derivative to find $f'(x)$ if

$$f(x) = \frac{2x+1}{x+2}$$

Note: no marks will be given if you do not use the limit definition of the derivative.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{2(x+h)+1}{(x+h)+2} - \frac{2x+1}{x+2}}{h} = \lim_{h \to 0} \frac{\frac{2x+2h+1}{x+h+2} - \frac{2x+1}{x+2}}{h}$$

$$= \lim_{h \to 0} \frac{(x+2)(2x+2h+1) - (2x+1)(x+h+2)}{h(x+2)(x+h+2)}$$

$$= \lim_{h \to 0} \frac{\cancel{2x^2} + \cancel{2xh} + \cancel{x} + \cancel{4x} + 4h + \cancel{2} - \cancel{2x^2} - \cancel{2xh} - \cancel{4x} - \cancel{x} - h - \cancel{2}}{h(x+2)(x+h+2)}$$

$$= \lim_{h \to 0} \frac{3\cancel{h}}{\cancel{h}(x+2)(x+h+2)} = \lim_{h \to 0} \frac{3}{(x+2)(x+h+2)}$$

$$= \boxed{\frac{3}{(x+2)^2}}$$

5. (12 marks)

Find the derivatives of the following functions. Answers do not need to be simplified.

(a) $f(x) = \dfrac{\tan(x^3+1)}{\sqrt{x+\sqrt{x}}}$

$$f'(x) = \boxed{\dfrac{\sec^2(x^3+1)\cdot(3x^2)\cdot\sqrt{x+\sqrt{x}} - \tan(x^3+1)\cdot\dfrac{1}{2\sqrt{x+\sqrt{x}}}\left(1+\dfrac{1}{2\sqrt{x}}\right)}{\left(\sqrt{x+\sqrt{x}}\right)^2}}$$

(b) $f(x) = \csc^2\left(\sqrt{\sin(\sqrt[3]{x+1})}\right)$

$$f'(x) = \boxed{2\csc\left(\sqrt{\sin(\sqrt[3]{x+1})}\right)\cdot\left(-\csc\left(\sqrt{\sin(\sqrt[3]{x+1})}\right)\cot\left(\sqrt{\sin(\sqrt[3]{x+1})}\right)\right) \cdot \frac{1}{2\sqrt{\sin(\sqrt[3]{x+1})}}\cdot\cos\left(\sqrt[3]{x+1}\right)\cdot\frac{1}{3}(x+1)^{-2/3}\cdot(1)}$$

5. (continued)

(c) $f(x) = \dfrac{\arctan(x^3)}{\ln(x^6+1)}$

$$f'(x) = \boxed{\dfrac{\dfrac{1}{1+(x^3)^2}(3x^2)\cdot\ln(x^6+1) - \arctan(x^3)\cdot\dfrac{1}{x^6+1}(6x^5)}{\left(\ln(x^6+1)\right)^2}}$$

6. (4 marks)

If $F(x) = f\left(x^2 g(h(x))\right)$, where $g(1) = 5$, $g'(1) = 3$, $h(2) = 1$, $h'(2) = -2$, and $f'(20) = 2$, find $F'(2)$

$$F'(x) = f'(x^2 g(h(x))) \cdot \left(2x\, g(h(x)) + x^2 g'(h(x)) \cdot h'(x)\right)$$

$$F'(2) = f'(4g(h(2))) \cdot \left(4g(h(2)) + 4g'(h(2)) \cdot h'(2)\right)$$

$$= f'(4g(1)) \cdot \left(4g(1) + 4g'(1) \cdot (-2)\right)$$

$$= f'(4(5)) \cdot \left(4(5) + 4(3)(-2)\right)$$

$$= f'(20) \cdot (20 - 24) = (2)(-4) = \boxed{-8}$$

7. (10 marks)

(a) Find $\frac{dy}{dx}$ for the equation $\cos(xy) = \sin(x+y) + 3y^2$

$$-\sin(xy)\cdot\left(y + x\frac{dy}{dx}\right) = \cos(x+y)\cdot\left(1+\frac{dy}{dx}\right) + 6y\frac{dy}{dx}$$

$$-y\sin(xy) - x\sin(xy)\frac{dy}{dx} = \cos(x+y) + \cos(x+y)\frac{dy}{dx} + 6y\frac{dy}{dx}$$

$$-y\sin(xy) - \cos(x+y) = x\sin(xy)\frac{dy}{dx} + \cos(x+y)\frac{dy}{dx} + 6y\frac{dy}{dx}$$

$$-\left(y\sin(xy) + \cos(x+y)\right) = \left(x\sin(xy) + \cos(x+y) + 6y\right)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \boxed{-\frac{y\sin(xy) + \cos(x+y)}{x\sin(xy) + \cos(x+y) + 6y}}$$

(b) Find $\frac{dy}{dx}$ for the equation $x^{2y} = y^{2x}$

$$\ln(x^{2y}) = \ln(y^{2x})$$

$$2y\ln x = 2x\ln y$$

$$2\frac{dy}{dx}\ln x + 2y\left(\frac{1}{x}\right) = 2\ln y + 2x\left(\frac{1}{y}\right)\frac{dy}{dx}$$

$$\cancel{2}\ln x\frac{dy}{dx} - \frac{\cancel{2}x}{y}\frac{dy}{dx} = \cancel{2}\ln y - \frac{\cancel{2}y}{x}$$

$$\left(\ln x - \frac{x}{y}\right)\frac{dy}{dx} = \ln y - \frac{y}{x}$$

$$\frac{dy}{dx} = \frac{\ln y - \frac{y}{x}}{\ln x - \frac{x}{y}} = \boxed{\frac{xy\ln y - y^2}{xy\ln x - x^2}}$$

8. (4 marks)

Find the absolute maximum and absolute minimum of the function

$$f(x) = 3x^4 - 4x^3 - 12x^2 + 5$$

on the interval $[-2, 1]$

$$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x^2 - x - 2)$$
$$= \underline{12x(x-2)(x+1)}$$

$f'(x) = 0$ if $\underline{x=0}$, $\underline{x=2}$ or $\underline{x=-1}$

$x=2$ is not in $[-2,1]$, so we do not consider it.

$f(-1) = 3(-1)^4 - 4(-1)^3 - 12(-1)^2 + 5 = 3 + 4 - 12 + 5 = \boxed{0}$

$f(0) = 3(0)^4 - 4(0)^3 - 12(0)^2 + 5 = \boxed{5}$

endpoints:

$f(-2) = 3(-2)^4 - 4(-2)^3 - 12(-2)^2 + 5 = 48 + 32 - 48 + 5 = \boxed{37}$

$f(1) = 3(1)^4 - 4(1)^3 - 12(1)^2 + 5 = 3 - 4 - 12 + 5 = \boxed{-8}$

Absolute Maximum is 37, at $x = -2$

Absolute Minimum is -8, at $x = 1$

9. (14 marks)

$$\text{Let} \quad f(x) = \frac{x^3}{x-4}$$

(i) find the y- and x-intercepts of $f(x)$ (if any)

(ii) find the horizontal and vertical asymptotes of $f(x)$ (if any)

(iii) find the intervals where $f(x)$ is increasing and where $f(x)$ is decreasing

(iv) find the relative maxima and minima of $f(x)$ (if any)

(v) find the intervals where $f(x)$ is concave up and where $f(x)$ is concave down

(vi) find the inflection points of $f(x)$ (if any)

(vii) sketch the graph of $f(x)$

Note: $f'(x) = \dfrac{2x^2(x-6)}{(x-4)^2}$, $\quad f''(x) = \dfrac{2x(x^2-12x+48)}{(x-4)^3}$

(i) $f(0) = \dfrac{(0)^3}{0-4} = \dfrac{0}{-4} = 0$: y-intercept is $\boxed{(0,0)}$

$f(x) = 0$ if $\dfrac{x^3}{x-4} = 0$: $x^3 = 0$, $x = 0$

x-intercept is $\boxed{(0,0)}$

(ii) $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{x^3}{x-4} \cdot \dfrac{\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)} = \lim\limits_{x\to\infty} \dfrac{x^2}{1-\frac{4}{x}} \to \dfrac{\infty}{1-0} \to \infty$

$\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} \dfrac{x^3}{x-4} \cdot \dfrac{\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)} = \lim\limits_{x\to-\infty} \dfrac{x^2}{1-\frac{4}{x}} \to \dfrac{\infty}{1-0} \to \infty$

so there are no horizontal asymptotes.

Vertical asymptote at $\boxed{x=4}$

$\lim\limits_{x\to4^-} f(x) \to \dfrac{(4)^3}{0^-} \to \boxed{-\infty}$, $\quad \lim\limits_{x\to4^+} f(x) \to \dfrac{(4)^3}{0^+} \to \boxed{\infty}$

9. (continued)

(iii) $f'(x) = \dfrac{2x^2(x-6)}{(x-4)^2}$, so $f'(x)=0$ if $\underline{x=0}$ or $\underline{x=6}$

$f'(x)$ does not exist if $\underline{x=4}$

Interval	$2x^2$	$x-6$	$(x-4)^2$	$f'(x)$	f
$x<0$	+	−	+	−	↘
$0<x<4$	+	−	+	−	↘
$4<x<6$	+	−	+	−	↘
$x>6$	+	+	+	+	↗

f is increasing on $(6,\infty)$ and decreasing on $(-\infty,0)$, $(0,4)$ and $(4,6)$.

(iv) local minimum at $x=6$: $f(6) = \dfrac{6^3}{6-4} = 108$

so minimum at $\boxed{(6,108)}$

(v) $f''(x) = \dfrac{2x(x^2-12x+48)}{(x-4)^3}$

For $x^2-12x+48$, $b^2-4ac = (-12)^2-4(1)(48) = 144-192$

$= -48 < 0$

so $f''(x)=0$ if $\underline{x=0}$ and $f''(x)$ doesn't exist if $\underline{x=4}$

Interval	$2x$	$x^2-12x+48$	$(x-4)^3$	$f''(x)$	f
$x<0$	−	+	−	+	∪
$0<x<4$	+	+	−	−	∩
$x>4$	+	+	+	+	∪

f is concave up on $(-\infty,0)$ and $(4,\infty)$ and concave down on $(0,4)$

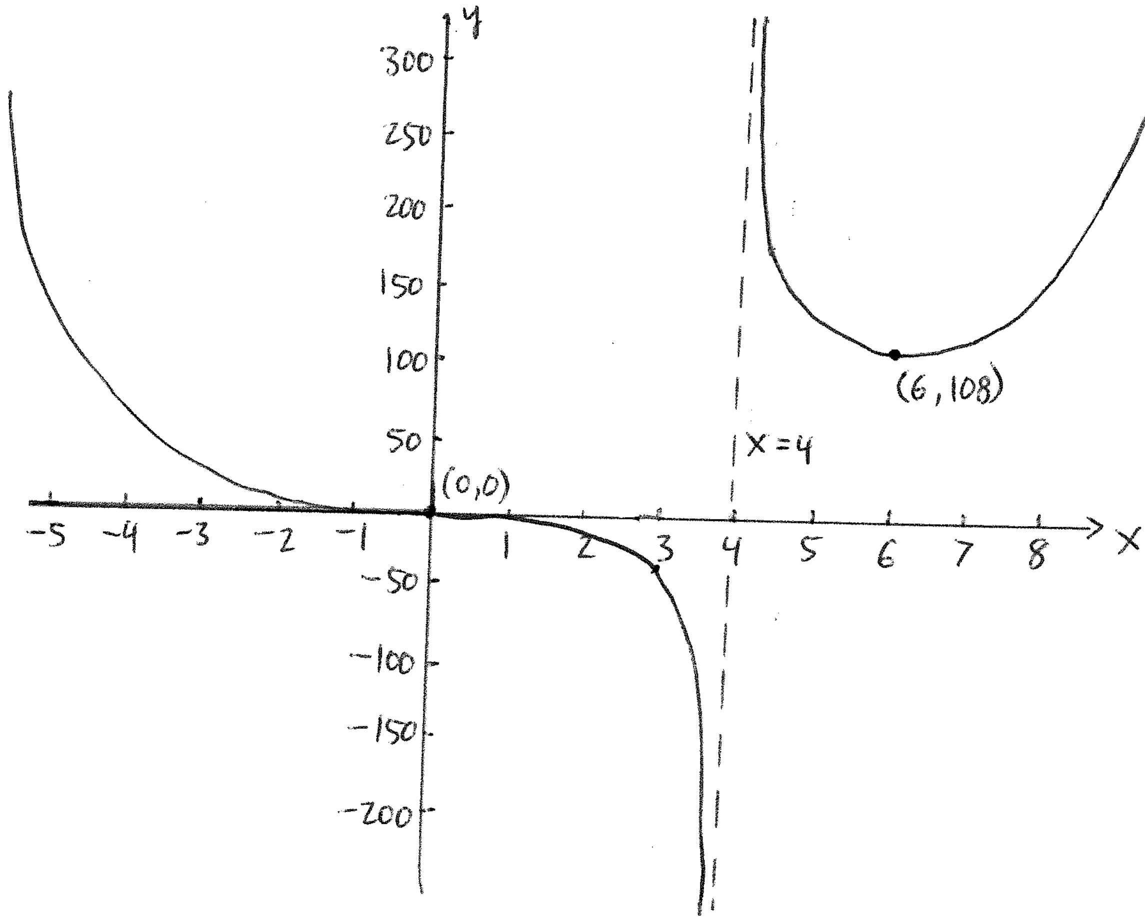
9. (continued)

(vi) inflection point at $x=0$: $f(0) = \frac{(0)^3}{0-4} = \frac{0}{-4} = 0$

$\boxed{(0,0)}$

there is no inflection point at $x=4$ since f is not defined there.

(vii)



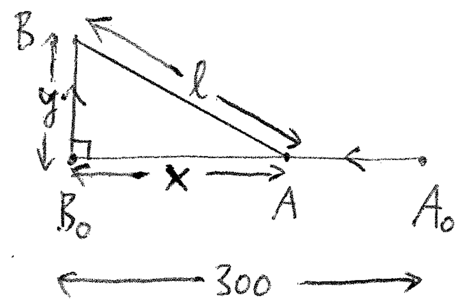
10. (5 marks)

At a given moment, Ship A is 300 km due east of Ship B. Ship A is sailing west at a constant speed of 60 km/h and Ship B is sailing north at a constant speed of 30 km/h. At what rate is the distance between the ships changing three hours later ?

Let x be the distance (in km) between Ship A and the initial position of Ship B.

Let y be the distance (in km) travelled by Ship B.

Let ℓ be the distance (in km) between the two ships.



$\frac{dx}{dt} = -60$ km/h , $\frac{dy}{dt} = 30$ km/h , $\frac{d\ell}{dt}$ is to be determined.

$$x^2+y^2=\ell^2$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2\ell\frac{d\ell}{dt}, \quad \text{so} \quad \frac{d\ell}{dt} = \frac{x}{\ell}\frac{dx}{dt} + \frac{y}{\ell}\frac{dy}{dt}$$

After three hours, Ship A has travelled $3(60) = 180$ km, so $x = 300-180 = \boxed{120}$. Ship B has travelled $3(30) = 90$ km so $\boxed{y = 90}$, so $\ell = \sqrt{(120)^2+(90)^2} = \boxed{150}$

$$\frac{d\ell}{dt} = \frac{120}{150}(-60) + \frac{90}{150}(30) = -48+18 = \boxed{-30 \text{ km/h}}$$

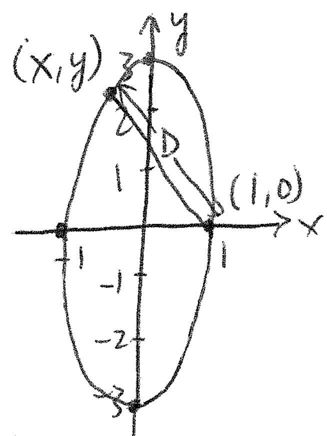
The distance between the ships is decreasing at a rate of 30 km/h.

11. (5 marks)

Find the points on the ellipse $9x^2 + y^2 = 9$ that are farthest away from the point $(1, 0)$

$x^2 + \frac{y^2}{9} = 1$, so $x^2 + \left(\frac{y}{3}\right)^2 = 1$

Let (x,y) be a point on the ellipse and let D be the distance between the points $(1,0)$ and (x,y)



$D = \sqrt{(x-1)^2 + y^2}$

Since (x,y) lies on the ellipse, $9x^2 + y^2 = 9$,

$\boxed{y^2 = 9 - 9x^2}$, so

$D = \sqrt{(x-1)^2 + 9 - 9x^2}$

To maximize D, we must maximize $\boxed{D^2 = (x-1)^2 + 9 - 9x^2}$

$\frac{d}{dx}(D^2) = 2(x-1) - 18x = 2x - 2 - 18x = -16x - 2 = \boxed{-2(8x+1)}$

$\frac{d}{dx}(D^2) = 0$ if $8x + 1 = 0$: $\underline{x = -\frac{1}{8}}$

If $x < -\frac{1}{8}$, $8x < -1$, $8x+1 < 0$, $-2(8x+1) > 0$, $\frac{d}{dx}(D^2) > 0$

If $x > -\frac{1}{8}$, $8x > -1$, $8x+1 > 0$, $-2(8x+1) < 0$, $\frac{d}{dx}(D^2) < 0$

So D^2 (and therefore D) is maximized when $\boxed{x = -\frac{1}{8}}$

$y^2 = 9 - 9x^2 = 9 - 9\left(-\frac{1}{8}\right)^2 = 9 - \frac{9}{64} = \frac{567}{64}$

So $y = \pm \frac{\sqrt{567}}{8} = \pm \frac{9\sqrt{7}}{8}$

The points are $\boxed{\left(-\frac{1}{8}, \frac{9\sqrt{7}}{8}\right)}$ and $\boxed{\left(-\frac{1}{8}, -\frac{9\sqrt{7}}{8}\right)}$

12. (8 marks)

Evaluate the following indefinite integrals

(a) $\displaystyle\int \left(2\sec x \tan x + \frac{3}{x} - \frac{5}{\sqrt{1-x^2}}\right) dx$

$= \boxed{2\sec x + 3\ln|x| - 5\arcsin x + C}$

(b) $\displaystyle\int \frac{1}{\sin^2 x\,\sqrt{1+\cot x}}\, dx$

$u = 1 + \cot x$, so $du = -\csc^2 x\, dx$

$$\int \frac{1}{\sin^2 x \sqrt{1+\cot x}}\, dx = \int \frac{\csc^2 x}{\sqrt{1+\cot x}}\, dx = \int \frac{1}{\sqrt{u}} (-du)$$

$$= -\int u^{-1/2}\, du = -\frac{u^{1/2}}{\left(\frac{1}{2}\right)} + C = -2\sqrt{u} + C$$

$$= \boxed{-2\sqrt{1+\cot x} + C}$$

13. (4 marks)

Show that the function $y = e^{3x}\cos(2x)$ satisfies the differential equation $y'' - 6y' + 13y = 0$

$$y = \boxed{e^{3x}\cos(2x)}$$

$$y' = \boxed{3e^{3x}\cos(2x) - 2e^{3x}\sin(2x)}$$

$$y'' = 9e^{3x}\cos(2x) - 6e^{3x}\sin(2x) - 6e^{3x}\sin(2x) - 4e^{3x}\cos(2x)$$

$$= \boxed{5e^{3x}\cos(2x) - 12e^{3x}\sin(2x)}$$

left side $= y'' - 6y' + 13y$

$$= (5e^{3x}\cos(2x) - 12e^{3x}\sin(2x))$$
$$- 6(3e^{3x}\cos(2x) - 2e^{3x}\sin(2x)) + 13e^{3x}\cos(2x)$$

$$= \cancel{5e^{3x}\cos(2x)} - \cancel{12e^{3x}\sin(2x)} - \cancel{18e^{3x}\cos(2x)}$$
$$+ \cancel{12e^{3x}\sin(2x)} + \cancel{13e^{3x}\cos(2x)}$$

$= 0 =$ right side ✓

14. (4 marks)

Find the solution of the differential equation $\frac{dy}{dx} = 6x^2e^y$ that satisfies the condition $y(0) = 0$

$$e^{-y}\,dy = 6x^2dx$$

$$\int e^{-y}\,dy = \int 6x^2dx$$

$$-e^{-y} = 2x^3 + C$$

$$e^{-y} = -2x^3 - C$$

$$e^{y} = -\frac{1}{2x^3+C}$$

$$y = \ln\left(-\frac{1}{2x^3+C}\right)$$

$y(0)=0$, so $y=0$ when $x=0$

$0 = \ln\left(-\frac{1}{C}\right)$, so $-\frac{1}{C} = 1$, $\boxed{C=-1}$

$$y = \ln\left(-\frac{1}{2x^3-1}\right)$$

$$\boxed{y = \ln\left(\frac{1}{1-2x^3}\right)}$$