DAWSON COLLEGE

Department of Mathematics

SOLUTIONS
VERSION 1

Final Examination

Calculus I
(Science)
201-NYA-05

Fall 2019

December 11, 2019

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Note:

- Write your answer in the space provided for each question in this examination paper.
- Only Sharp EL-531X, XG or XT calculators are allowed
- Use the reverse sides if needed, or for rough work (indicate this clearly).
- The exam has 19 pages (16 questions). It must be returned intact.

Reserved for Marking

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	Q12	Q13	Q14	Q15	Q16
/12	/6	/4	/3	/12	/4	/3	/5	/5	/5	/3	/8	/15	/5	/6	/4

1. **[12 marks]** Find the limits, and if it doesn't exist, explain why (sided limits). Do not use $L'H\hat{o}pital$'s rule:

(a) $\lim_{x \to 0} \frac{\sqrt{16+5x}-4}{3x}$ $= \lim_{x \to 0} \frac{5x}{3x(\sqrt{16+5x}+4)} = \frac{5}{24}$

(b) $\lim_{x \to 5^+} (x^2-25)\cos\left(\frac{1}{x-5}\right)$

$-1 \leq \cos\left(\frac{1}{x-5}\right) \leq 1$

$-(x^2-25) \leq (x^2-25)\cos\left(\frac{1}{x-5}\right) \leq (x^2-25)$

Since $\lim_{x \to 5^+} -(x^2-25) = \lim_{x \to 5^+} (x^2-25) = 0$

by Squeeze Theorem, $\lim_{x \to 5^+} (x^2-25)\cos\left(\frac{1}{x-5}\right) = 0$

(c) $\lim_{x\to 4} \frac{x^2 - x - 12}{|\, x - 4 \,|}$ d.n.e

$\downarrow$

$$\lim_{x\to 4^-} \frac{x^2-x-12}{|x-4|} = \lim_{x\to 4^-} \frac{\cancel{(x-4)}(x+3)}{-\cancel{(x-4)}} = -7$$

$$\lim_{x\to 4^+} \frac{x^2-x-12}{|x-4|} = \lim_{x\to 4^+} \frac{\cancel{(x-4)}(x+3)}{\cancel{x-4}} = 7$$

(d) $\lim_{x\to\infty} \arctan\left(\frac{1-x^2}{x\sqrt{x}}\right) = \arctan \lim_{x\to\infty} \left(\frac{1-x^2}{x\sqrt{x}}\right) = -\frac{\pi}{2}$

2. **[6 marks]** Given the function f:

$$f(x) = \begin{cases} \dfrac{2x^2+3x+1}{x+1} & , \text{ if } x < 1 , \\ 3 & , \text{ if } x = 1 , \\ \dfrac{3x-1}{2-x} & , \text{ if } x > 1 . \end{cases}$$

Find the x-values where f is discontinuous and specify the type of discontinuity in each case. Refer to the definition of continuity to justify.

- $x = 1$ JUMP DISCONTINUITY

because $\lim_{x \to 1} f(x)$ d.n.e. $\begin{cases} \lim_{x \to 1^-} f(x) = 3 \\ \lim_{x \to 1^+} f(x) = 2 \end{cases}$

- $x = -1$ HOLE (REMOVABLE) DISCONTINUITY

because $f(-1)$ d.n.e, $\lim_{x \to -1} f(x) = \lim_{x \to -1} \dfrac{\cancel{(x+1)}(2x+1)}{\cancel{x+1}} = -1$

- $x = 2$ INFINITE DISCONTINUITY

because $f(2)$ d.n.e $\lim_{x \to 2} f(x)$ d.n.e. $\begin{cases} \lim_{x \to 2^-} \dfrac{3x-1}{2-x} = +\infty \\ \lim_{x \to 2^+} \dfrac{3x-1}{2-x} = -\infty \end{cases}$

3. **[4 marks]** Let $f(x) = \dfrac{5}{3x+7}$. Find $f'(x)$ using only the limit definition of the derivative.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{5}{3(x+h)+7} - \frac{5}{3x+7}}{h}$$

$$= \lim_{h\to 0} \frac{15x+35-15x-15h-35}{[3(x+h)+7](3x+7)\cdot h} = \frac{-15}{(3x+7)^2}$$

4. **[3 marks]** Given $f(x) = | 2x - 1 |$. Determine if it is differentiable and, if not, find the point (s) where it is not.

f is not differentiable at $x = \frac{1}{2}$, differentiable everywhere else on $\mathbb{R}$

5. **[12 marks]** Find the derivatives for the following functions; do not simplify.

(a) $y = \dfrac{\sqrt{3x-4}}{(5x+2)^5}$

$$y' = \frac{\frac{1}{2\sqrt{3x-4}} \cdot 3 \cdot (5x+2)^5 - \sqrt{3x-4} \cdot 5(5x+2)^4 \cdot 5}{\left((5x+2)^5\right)^2}$$

(b) $y = \arctan(\pi 5^{3x}) sec(4x) + \cos^2\left(\dfrac{1}{\csc x}\right)$

$$y' = \frac{1}{1+(\pi 5^{3x})^2} \cdot \pi \cdot 5^{3x} \ln 5 \cdot 3 \sec(4x) + \arctan(\pi 5^{3x}) \cdot$$

$$\cdot \sec(4x) \tan(4x) \cdot 4 + 2\cos\left(\frac{1}{\csc x}\right) \cdot \left(-\sin\left(\frac{1}{\csc x}\right)\right) \left(-\frac{1}{\csc^2 x}\right) \cdot$$

$$\cdot (-\csc x)(\cot x)$$

(c) $y = 5e^5 + \ln(ln(x)) - \log_5 \frac{5}{x}$

$$y' = \frac{1}{\ln x} \cdot \frac{1}{x} + \frac{1}{x \ln 5}$$

6. **[4 marks]** Given $f(x) = x^{\frac{1}{x}}$ find $f''(x)$.

$$y = x^{\frac{1}{x}}$$

$$\ln y = \frac{1}{x} \ln x \qquad \frac{y'}{y} = -\frac{1}{x^2} \ln x + \frac{1}{x^2}$$

$$y' = \frac{1 - \ln x}{x^2} \cdot x^{\frac{1}{x}}$$

$$y'' = \frac{-\frac{1}{x} \cdot x^2 - (1 - \ln x) \cdot 2x}{x^4} \cdot x^{\frac{1}{x}} + \frac{1 - \ln x}{x^2} \cdot \frac{1 - \ln x}{x^2} \cdot x^{\frac{1}{x}}$$

7. **[3 marks]** A thermometer is moved from inside a house out to the deck. Its temperature (in degrees Fahrenheit) t minutes after it has been moved is given by

$$T(t) = 30 + 40e^{-0.98t}$$

(a) What is the temperature inside the house?

$T(0) = 30 + 40e^{-0.98\cdot 0} = 70$ (degrees F)

(b) How fast is the reading on the thermometer changing 3 minutes after it has been taken out of the house?

$T'(t) = 40\cdot e^{-0.98t}\cdot(-0.98)$

$T'(3) \simeq -2.07$ (degrees F / min)

(c) Find $\lim_{t\to\infty} T(t)$ and interpret the result.

$\lim_{t\to\infty}(30+40e^{-0.98t}) = 30$

After a long time outside the temperature will read 30 °F.

This is the temperature outside on the deck.

8. **[5 marks]** Given $f(x) = 2\arctan x$ and $g(x) = \ln(1+x^2) + \dfrac{\pi}{2} - \ln 2$, find whether they have any common tangent line at one and the same point, and if yes, find the equation of the tangent(s).

$$f'(x) = g'(x)$$

$$\frac{2}{1+x^2} = \frac{2x}{1+x^2}$$

$$2x = 2$$

$$x = 1$$

$$f(1) = g(1) = \frac{\pi}{2}$$

Point of common tangent $\left(1, \frac{\pi}{2}\right)$

Slope of the tangent : $f'(1) = g'(1) = 1$

Equation of the tangent

$$y = ax + b$$

$$\frac{\pi}{2} = 1 + b$$

$$b = \frac{\pi}{2} - 1$$

$$\boxed{y = x + \frac{\pi}{2} - 1}$$

9. **[5 marks]** Find the equation of the tangent line to the graph of the relation

$$\ln(xy) = y^2 - 1$$

at the point $(1, 1)$.

$$\frac{y + xy'}{xy} = 2yy'$$

Replace $(1,1)$

$$1 + y'\big|_{(1,1)} = 2y'\big|_{(1,1)}$$

$$y'\big|_{(1,1)} = 1$$

$$y = ax + b$$

$$1 = 1 \cdot 1 + b$$

$$b = 0$$

$$\boxed{y = x}$$

10. **[5 marks]** A particle is moving along the curve $y = 2cos\,(\pi x)$. As the particle passes through the point of x coordinate $\frac{1}{4}$, its x coordinate increases at a rate of 2 cm/s. How fast is the distance from the particle to the origin changing at this instant (i.e., when the x-coordinate is $\frac{1}{4}$)?

$$Y = 2\cos(\pi x)$$

$$\frac{dy}{dt} = -2\sin(\pi x)\cdot\pi\cdot\frac{dx}{dt}$$

$$\frac{dx}{dt} = 2\ [cm/s]$$

$$x = \frac{1}{4} \to y = \sqrt{2}$$

$$z = \sqrt{\frac{1}{16} + 2} = \frac{\sqrt{33}}{4}$$

$$z^2 = x^2 + y^2$$

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\frac{dz}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{z} = \frac{\frac{1}{4}\cdot 2 + \sqrt{2}\cdot\left(-2\sin\frac{\pi}{4}\right)\cdot\pi\cdot 2}{\frac{\sqrt{33}}{4}}$$

$$\approx -8.402\ [cm/s]$$

11. **[3 marks]** Indicate whether each of the following statements is true or false. Justify or give a counterexample.

(a) Rational functions can only have infinite discontinuities (vertical asymptotes).

False, counterexample: $f(x) = \frac{(x-2)(x+3)}{(x-2)(x+4)}$

is not cont. at $x=2$, but $x=2$ is a removable discontinuity

(b) Polynomial functions have absolute extrema on any closed interval.

True: Polynomial functions are continuous, and continuous functions have absolute extrema on closed intervals.

(c) Any inflection point is also a critical point.

False, counterexample $f(x) = \tan x$ at $x=0$ there's an inflection point, but the slope is 1.

12. **[8 marks]** Find the limits. Use *L'Hôpital*'s rule, if applicable:

(a) $\lim_{x \to 0} \left(\frac{1}{\ln(x+1)} - \frac{1}{x} \right)$ $\infty - \infty$

$$= \lim_{x \to 0} \frac{x - \ln(x+1)}{x \ln(x+1)} \underset{L'H}{\overset{\frac{0}{0}}{=}} \lim_{x \to 0} \frac{1 - \frac{1}{x+1}}{\frac{x}{x+1} + \ln(x+1)}$$

$$\underset{L'H}{\overset{\frac{0}{0}}{=}} \lim_{x \to 0} \frac{\frac{1}{(x+1)^2}}{\frac{x+1-x}{(x+1)^2} + \frac{1}{x+1}} = \frac{1}{2}$$

(b) $\lim_{x \to 0^+} e^{\frac{1}{x}} \ln x$ $= -\infty$

(c) $\lim_{x\to 0^+} (2^x - 1)^x$ $\quad 0^0$

$$= e^{\ln \lim_{x\to 0^+} (2^x-1)^x} = e^{\lim_{x\to 0^+} x \ln(2^x-1)} = e^L$$

$$L = \lim_{x\to 0^+} \frac{\ln(2^x-1)}{\frac{1}{x}} \overset{\frac{\infty}{\infty}}{\underset{L'H}{=}} \lim_{x\to 0^+} \frac{\frac{2^x \ln 2}{2^x-1}}{-\frac{1}{x^2}}$$

$$= -\ln 2 \lim_{x\to 0^+} \frac{2^x x^2}{2^x - 1} \overset{\frac{0}{0}}{\underset{L'H}{=}} -\ln 2 \lim_{x\to 0^+} \frac{\cancel{2^x} \ln 2\, x^2 + \cancel{2^x} \cdot 2x}{\cancel{2^x} \ln 2} =$$

$$= -\ln 2 \cdot 0 = 0$$

$$\lim_{x\to 0^+} (2^x - 1)^x = e^0 = 1$$

13. **[15 marks]** Consider the function:

$$f(x) = \frac{(1+x)^4}{(1-x)^4} \text{ with } f'(x) = \frac{8(1+x)^3}{(1-x)^5} \text{ and } f''(x) = \frac{16(1+x)^2(x+4)}{(1-x)^6}.$$

(a) Find its x- and y- intercepts.

x-int. $y=0 \Rightarrow x=-1$ $(-1,0)$

y-int: $x=0 \Rightarrow y=1$ $(0,1)$

(b) Find its asymptotes, if any. Justify your answer using limits.

Vertical $x=1$ $\lim_{x\to 1} \frac{(1+x)^4}{(1-x)^4} = +\infty$

Horizontal $y=1$ $\lim_{x\to\pm\infty} \frac{(1+x)^4}{(1-x)^4} = 1$

(c) Find the intervals where f is increasing and where f is decreasing, and local extrema, if any.

$f'(x)=0 \quad x=-1$ critical point

x	$-\infty$		-1		1		$+\infty$
f'		$-$	0	$+$	‖	$-$	
f	1	$\searrow$	0	$\nearrow$ $+\infty$	‖	$+\infty$ $\searrow$	1

f increasing: $x \in (-1, 1)$

local min: $(-1, 0)$

f decreasing $x \in (-\infty; -1) \cup (1, +\infty)$

(d) Find the intervals where f is convex (concave upward) and where f is concave (concave downward), and points of inflection, if any.

$f''(x)=0 \quad x=-1$ and $x=-4$ potential points of inflection

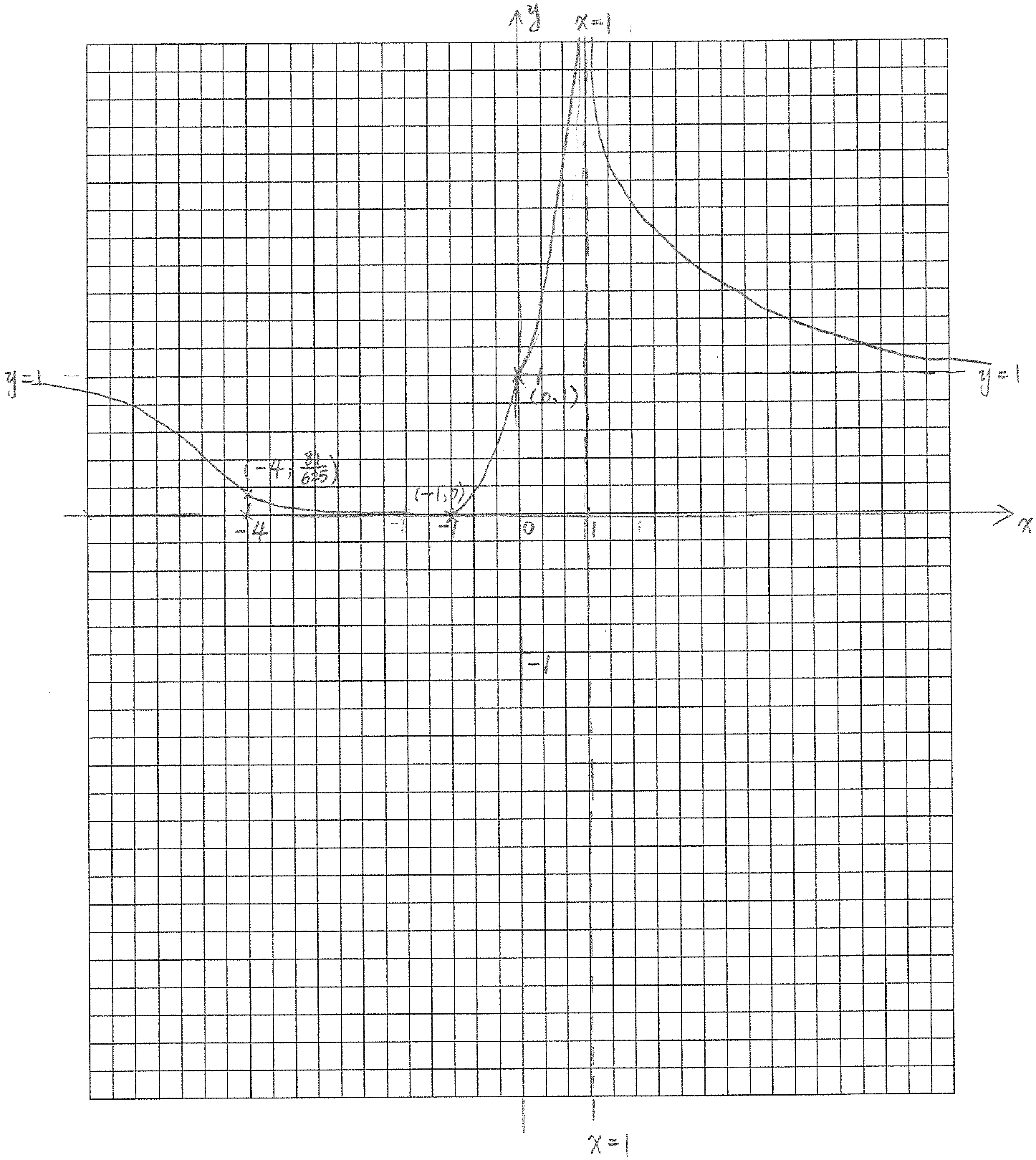
x	$-\infty$		-4		-1		1		$+\infty$
f''		$-$	0	$+$	0	$+$	‖	$+$	
f		$\frown$	$\frac{81}{625} = 0.1296$	$\smile$	0	$\smile$	‖	$\smile$	

f convex: $x \in (-4, -1) \cup (-1, 1) \cup (1, +\infty)$

f concave: $x \in (-\infty; -4)$

Point of inflection $\left(-4, \frac{81}{625}\right)$

(e) Graph the function clearly labeling all important points.



14. **[5 marks]** The designer of a book wants its pages to have 1-in margins at the top and bottom and 0.5-in margins on the side. He also wants each page to have an area of 50 in^2. Determine the page dimensions that will result in the maximum printed area on the page.

$$\begin{cases} xy = 50 & \rightarrow y = \frac{50}{x} \\ A = (x-2)(y-1) & \end{cases}$$

$$A = (x-2)\left(\frac{50}{x} - 1\right) = 50 - x - \frac{100}{x} + 2$$

$$A' = -1 + \frac{100}{x^2} = 0$$

$$x = 10 \quad y = \frac{50}{10} = 5$$

$$A'' = \frac{-200}{x^3} \quad A''(10) < 0 \rightarrow \text{so } x = 10 \text{ gives max } A$$

15. **[6 marks]** Find the antiderivatives:

(a) $\int 3x^2 \left(\frac{4}{x^3} + \frac{5}{\sqrt{x}} + 8x^7 \right) dx$

$= \int \left(12x^{-1} + 15x^{3/2} + 24x^9\right) dx$

$= 12 \ln|x| + 6x^{5/2} + \frac{12}{5}x^{10} + C$

(b) $\int (\sin x \cos x) e^{\sin^2 x} dx \quad = \frac{1}{2} e^{\sin^2 x} + C$

16. **[4 marks]** Given $f''(\theta) = \sin\theta - \cos\theta$ and $f(0) = 2$ and $f'(0) = 4$, find $f(\theta)$.

$$f'(\theta) = \int (\sin\theta - \cos\theta)\, d\theta\, d\theta = -\cos\theta - \sin\theta + C_1$$

$$f'(0) = 0 \Rightarrow C_1 = 5$$

$$f(\theta) = \int (-\cos\theta - \sin\theta + 5)\, d\theta = -\sin\theta + \cos\theta + 5\theta + C_2$$

$$f(0) = 0 \Rightarrow C_2 = 1$$

So $f(\theta) = -\sin\theta + \cos\theta + 5\theta + 1$