

DAWSON COLLEGE
MATHEMATICS DEPARTMENT

FINAL EXAMINATION
FALL 2025

Linear Algebra (201-NYC-05)

Examiners: Gorelyshev I., Ohanyan V.

Student Name: Solution

Student I.D.: _____

Date: December 16, 2025

Time: 3 Hours

Instructions:

- Print your name and student ID number in the space provided above.
- All questions are to be answered directly on the examination paper in the space provided.
- Small, noiseless and non-programmable calculators are permitted.

This examination consists of 14 questions.

Please ensure that you have a complete examination before starting.

This exam must be returned intact.

1. [6+3 marks]

a) Use the Gauss-Jordan elimination method to find the general solution of the following system of linear equations:

$$\begin{cases} x - y + z - 4w = 0, \\ x + 2y + z - w = 6, \\ 2x + y - 2z + 7w = 2. \end{cases}$$

$$\left(\begin{array}{cccc|c} 1 & -1 & 1 & -4 & 0 \\ 1 & 2 & 1 & -1 & 6 \\ 2 & 1 & -2 & 7 & 2 \end{array} \right) \Rightarrow \left(\begin{array}{cccc|c} 1 & -1 & 1 & -4 & 0 \\ 0 & 3 & 0 & 3 & 6 \\ 0 & 3 & -4 & 15 & 2 \end{array} \right) \Rightarrow \left(\begin{array}{cccc|c} 1 & -1 & 1 & -4 & 0 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 3 & -4 & 15 & 2 \end{array} \right)$$

$$\Rightarrow \left(\begin{array}{cccc|c} 1 & -1 & 1 & -4 & 0 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & -4 & 12 & -4 \end{array} \right) \Rightarrow \left(\begin{array}{cccc|c} 1 & -1 & 1 & -4 & 0 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & -3 & 1 \end{array} \right) \Rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & -3 & 1 \end{array} \right)$$

$$\underline{x=1, y=2-t, z=1+3t, w=t.}$$

b) Find the particular solution of the system in which $y = 1$.

$$y=1 \Rightarrow t=1 \Rightarrow \underline{x=1, y=1, z=4, w=1}$$

2. [7 marks] Determine the values of parameter k for which the following system has

a) one solution; b) infinitely many solutions.

$$\begin{cases} 8x + (3-k)y = 0, \\ (k-3)x - 2y = 0. \end{cases}$$

$$\begin{cases} x + ky + z = 0 \\ x + ky = 0 \\ kx + 9y + kz = 0 \end{cases} \Rightarrow \begin{pmatrix} 1 & k & 1 \\ 1 & k & 0 \\ k & 9 & k \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & k & 1 \\ 0 & 0 & -1 \\ 0 & 9 & k \end{pmatrix} \Rightarrow k^2 - 9 = \det(A)$$

a) $k \neq \pm 3$, b) $k = \pm 3$.

3. [7+3 marks] Given the system of linear equations

$$\begin{cases} x + 2z = 5, \\ 4x + y - z = 1, \\ 3x + 5z = 13 \end{cases} \Rightarrow \begin{pmatrix} 1 & 0 & 2 \\ 4 & 1 & -1 \\ 3 & 0 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 13 \end{pmatrix} \quad \det(A) = \begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix} = -1$$

a) Use the adjoint matrix to find the inverse of the coefficient matrix.

$$M = \begin{pmatrix} 5 & 23 & -3 \\ 0 & -1 & 0 \\ -2 & -9 & 1 \end{pmatrix} \Rightarrow C = \begin{pmatrix} 5 & -23 & -3 \\ 0 & -1 & 0 \\ -2 & 9 & 1 \end{pmatrix}$$

$$\text{adj}(A) = C^T = \begin{pmatrix} 5 & 0 & -2 \\ -23 & -1 & 9 \\ -3 & 0 & 1 \end{pmatrix}$$

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A) = \begin{pmatrix} -5 & 0 & 2 \\ 23 & 1 & -9 \\ 3 & 0 & -1 \end{pmatrix}$$

$$\text{check: } \begin{pmatrix} -5 & 0 & 2 \\ 23 & 1 & -9 \\ 3 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 4 & 1 & -1 \\ 3 & 0 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \checkmark$$

b) Use the inverse of the coefficient matrix to solve the system.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5 & 0 & 2 \\ 2 & 3 & 1 \\ 3 & 0 & -1 \end{pmatrix} \begin{pmatrix} 5 \\ 1 \\ 13 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \quad \checkmark$$

4. [7 marks] Use the Cramer's rule to solve the following system of linear equations:

$$\begin{cases} x + y - z = -2, \\ 2x - y + z = 5, \\ x + 3y + 2z = 7 \end{cases} \quad \underbrace{\begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \\ 1 & 3 & 2 \end{pmatrix}}_A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \\ 7 \end{pmatrix}$$

$$\det(A) = \begin{vmatrix} 1 & 1 & -1 \\ 0 & -3 & 3 \\ 0 & 2 & 3 \end{vmatrix} = -15$$

$$|A_x| = \begin{vmatrix} -2 & 1 & -1 \\ 5 & -1 & 1 \\ 7 & 3 & 2 \end{vmatrix} = \begin{vmatrix} -2 & 1 & -1 \\ 3 & 0 & 0 \\ 3 & 5 & 0 \end{vmatrix} = -15$$

$$|A_y| = \begin{vmatrix} 1 & -2 & -1 \\ 2 & 5 & 1 \\ 1 & 7 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -2 & -1 \\ 0 & 9 & 3 \\ 0 & 9 & 3 \end{vmatrix} = 0$$

$$|A_z| = \begin{vmatrix} 1 & 1 & -2 \\ 2 & -1 & 5 \\ 1 & 3 & 7 \end{vmatrix} = \begin{vmatrix} 1 & 1 & -2 \\ 0 & -3 & 9 \\ 0 & 2 & 9 \end{vmatrix} = -45$$

$$\underline{x = 1, \quad y = 0, \quad z = 3}$$

5. [7+3 marks] Given $A = \begin{pmatrix} 1 & -1 \\ 2 & 0 \\ -2 & 5 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$, find if possible

a) $\text{tr}(AB - I)$

$$AB = \begin{pmatrix} 1 & -1 \\ 2 & 0 \\ -2 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} -3 & -3 & -3 \\ 2 & 4 & 6 \\ 18 & 21 & 24 \end{pmatrix}$$

$$AB - I = \begin{pmatrix} -4 & -3 & -3 \\ 2 & 3 & 6 \\ 18 & 21 & 23 \end{pmatrix}$$

$$\text{tr}(AB - I) = -4 + 3 + 23 = 22$$

b) $\det(A^3 B)$

$A^3 B$ is undefined.

6. [5 marks] Find matrix X , if $(2X^T - I)^{-1} = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$.

$$2X^T - I = \frac{1}{-1} \begin{pmatrix} 5 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}$$

$$2X^T = \begin{pmatrix} -4 & 2 \\ 3 & 0 \end{pmatrix} \Rightarrow X^T = \begin{pmatrix} -2 & 1 \\ 3/2 & 0 \end{pmatrix}$$

$$\underline{X = \begin{pmatrix} -2 & 3/2 \\ 1 & 0 \end{pmatrix}}$$

7. [5 marks] Evaluate the determinant $\begin{vmatrix} 1 & -1 & 0 & 3 \\ 2 & 1 & 2 & 4 \\ 0 & 3 & 0 & 1 \\ 1 & 0 & -2 & -1 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 0 & 3 \\ 0 & 3 & 2 & -2 \\ 0 & 3 & 0 & 1 \\ 0 & 1 & -2 & -4 \end{vmatrix} =$

$$= \begin{vmatrix} 3 & 2 & -2 \\ 3 & 0 & 1 \\ 1 & -2 & -4 \end{vmatrix} = \begin{vmatrix} 9 & 2 & -2 \\ 0 & 0 & 1 \\ 13 & -2 & -4 \end{vmatrix} = - \begin{vmatrix} 9 & 2 \\ 13 & -2 \end{vmatrix} =$$

$$= -(-18 - 26) = \underline{44}$$

$$\text{OR } \begin{vmatrix} 1 & -10 & 0 & 3 \\ 2 & -11 & 2 & 4 \\ 0 & 0 & 0 & 1 \\ 13 & -2 & -1 & \end{vmatrix} = - \begin{vmatrix} 1 & -10 & 0 \\ 2 & -11 & 2 \\ 1 & 3 & -2 \end{vmatrix} = - \begin{vmatrix} 1 & -10 & 0 \\ 3 & -8 & 0 \\ 1 & 3 & -2 \end{vmatrix} = 2 \begin{vmatrix} 1 & -10 \\ 3 & -8 \end{vmatrix} = 44$$

$$\text{OR } \begin{vmatrix} 1 & -1 & 0 & 3 \\ 2 & 1 & 2 & 4 \\ 0 & 3 & 0 & 1 \\ 3 & 1 & 0 & 3 \end{vmatrix} = -2 \begin{vmatrix} 1 & -1 & 3 \\ 0 & 3 & 1 \\ 3 & 1 & 3 \end{vmatrix} = -2 \begin{vmatrix} 1 & -1 & 3 \\ 0 & 3 & 1 \\ 0 & 4 & -6 \end{vmatrix} =$$

$$= -2 \begin{vmatrix} 3 & 1 \\ 4 & -6 \end{vmatrix} = 44$$

8.[3+3 marks] Let A and B be 3x3 matrices such that $\det(A) = 5$, $\det(B) = -4$. Find

a) $\det(-3[AB]^T) = -27 |A| \cdot |B| = \underline{540}$

b) $\det(B^{-1} \text{adj}(A) + \text{adj}(B)A^{-1}) = \det(5B^{-1}A^{-1} - 4B^{-1}A^{-1}) = \det(B^{-1}A^{-1}) =$
 $\text{adj}(A) = \det(A) A^{-1}$
 $\text{adj}(B) = \det(B) B^{-1}$
 $= \det(B^{-1}) \det(A^{-1}) =$
 $= \frac{1}{\det(B)} \cdot \frac{1}{\det(A)} = \underline{-\frac{1}{20}}$

9.[3+3+3+3 marks] Let $\vec{u} = (1, 2, 3)$, $\vec{v} = (2, 0, 1)$, $\vec{w} = (0, 1, -1)$. Find

a) the cosine of the angle between $\vec{u}$ and $\vec{v}$,

$$\cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \cdot \|\vec{v}\|} = \frac{2+0+3}{\sqrt{14} \cdot \sqrt{5}} = \underline{\sqrt{\frac{5}{14}}}$$

b) a vector perpendicular to both $\vec{v}$ and $\vec{w}$,

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 1 \\ 0 & 1 & -1 \end{vmatrix} = -\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} = \underline{(-1, 2, 2)}$$

c) the area of the triangle determined by $\vec{u}$ and $\vec{w}$,

$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 0 & 1 & -1 \end{vmatrix} = -5\vec{i} + \vec{j} + \vec{k} = (-5, 1, 1)$$

$$A = \frac{1}{2} \|\vec{n}\| = \frac{1}{2} \sqrt{25+1+1} = \frac{\sqrt{27}}{2} = \frac{3\sqrt{3}}{2}$$

d) the volume of the parallelepiped determined by $\vec{u}$, $\vec{v}$ and $\vec{w}$.

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 0 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 5 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = - \begin{vmatrix} 1 & 5 \\ 2 & 1 \end{vmatrix} = 9$$

Ans: $V = 9$

10. (5 marks) Find the point-normal form of the equation of the plane containing the point $P(1,1,1)$ and the

line $\begin{cases} x = 1 + t \\ y = 1 - t \\ z = 2 + t \end{cases} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \vec{v}t, \quad \vec{v} = (1, -1, 1) \quad \Rightarrow \vec{u} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$\Rightarrow \vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{vmatrix} = \begin{vmatrix} \vec{i} & \vec{j} \\ 1 & -1 \end{vmatrix} = \vec{i} + \vec{j} = (1, 1, 0)$$

$$\Rightarrow (x-1) \cdot 1 + (y-1) \cdot 1 = 0 \Rightarrow x+y=2$$

11. (3 marks) Find the point-normal form of the equation of the plane passing through the point $P(5, 0, -4)$ and perpendicular to the line $x = 2 - 3t$, $y = 1 + t$, $z = 2t$.

$$= (2, 1, 0) + t \cdot \underbrace{(-3, 1, 2)}_{\vec{n}}$$

$$\underline{-3 \cdot (x - 5) + 1 \cdot (y - 0) + 2 \cdot (z + 4) = 0}$$

$$\underline{-3x + y + 2z = -23}$$

12. (3+3+3 marks) Given the line $\begin{cases} x = 2 - 3t \\ y = 1 + t \\ z = 2t \end{cases}$ and the plane $x + y + 3z = 7$, find

a) the point of intersection of the line and the plane,

$$2 - 3t + 1 + t + 6t = 7 \Rightarrow 4t = 4 \Rightarrow t = 1 \Rightarrow$$

$$\Rightarrow \underline{x = -1, y = 2, z = 2}$$

b) the distance from the point $A(2, 0, -1)$ to the plane,

$$D = \frac{|1 \cdot 2 + 1 \cdot 0 + 3 \cdot (-1) - 7|}{\sqrt{1^2 + 1^2 + 3^2}} = \underline{\underline{\frac{8}{\sqrt{11}}}}$$

c) the distance from the point $B(0,0,0)$ to the line.

$$\text{the line is } (x,y,z) = \underbrace{(2,1,0)}_P + t \underbrace{(-3,1,2)}_{\vec{v}}$$

$$\vec{u} = BP = (2,1,0)$$

$$A = \sqrt{4+16+25} = \sqrt{45}$$

$$\begin{vmatrix} i & j & k \\ 2 & 1 & 0 \\ -3 & 1 & 2 \end{vmatrix} = 2i - 4j + 5k = (2, -4, 5)$$

$$A = D \cdot \|\vec{v}\|$$

$$D = \frac{\sqrt{45}}{\sqrt{14}}$$

$$\|\vec{v}\| = \sqrt{9+1+4} = \sqrt{14}$$

13. [5 marks] Find the parametric equations of the line passing through the point $P(-1,2,2)$ and perpendicular to both lines $x=2-3t, y=1+t, z=2t$ and $x=s, y=-s, z=4$.

$$\vec{u} = (-3, 1, 2)$$

$$\vec{v} = (1, -1, 0)$$

$$\vec{n} \approx \begin{vmatrix} i & j & k \\ -3 & 1 & 2 \\ 1 & -1 & 0 \end{vmatrix} = +2i + 2j + 2k \Rightarrow (+1, 1, 1)$$

$$\text{Ans: } (x,y,z) = (-1, 2, 2) + r(1, 1, 1) = \begin{pmatrix} -1+r \\ 2+r \\ 2+r \end{pmatrix}$$

14. [7 marks] Maximize $P = 6x + 4y + 8z$ subject to:

$$\begin{cases} x + y + z \leq 20 \\ x + 2y + z \leq 60 \\ 2x + z \leq 10 \end{cases} \quad x, y, z \geq 0$$

$$\begin{aligned} x + y + z + s_1 &= 20 \\ x + 2y + z + s_2 &= 60 \\ 2x + z + s_3 &= 10 \end{aligned}$$

$$\left(\begin{array}{cccccc|c} 1 & 1 & 1 & 1 & 0 & 0 & 20 \\ 1 & 2 & 1 & 0 & 1 & 0 & 60 \\ 2 & 0 & 1 & 0 & 0 & 1 & 10 \\ \hline -6 & -4 & -8 & 0 & 0 & 0 & 0 \end{array} \right)$$

$$\Rightarrow \left(\begin{array}{cccccc|c} -1 & 1 & 0 & 1 & 0 & -1 & 10 \\ -1 & 2 & 0 & 0 & 1 & -1 & 50 \\ 2 & 0 & 1 & 0 & 0 & 1 & 10 \\ \hline 10 & -4 & 0 & 0 & 0 & 8 & 80 \end{array} \right) \Rightarrow \left(\begin{array}{cccccc|c} -1 & 1 & 0 & 1 & 0 & -1 & 10 \\ 1 & 0 & 0 & -2 & 1 & 1 & 30 \\ 2 & 0 & 1 & 0 & 0 & 1 & 10 \\ \hline 6 & 0 & 0 & 4 & 0 & 4 & 120 \end{array} \right)$$

$$P = 120 - 6x - 4s_1 - 4s_3 \Rightarrow \underline{P_{\max} = 120}$$

$$x = 0, s_1 = 0, s_3 = 0$$

$$y = 10, z = 10, s_2 = 30$$