DAWSON COLLEGE, MATHEMATICS DEPARTMENT, FINAL EXAMINATION

LINEAR ALGEBRA, 201-NYC-05 (01-09), MAY 27, 2025, Time: 3 hours

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1. (6+2 points) Given the matrix $A = \begin{pmatrix} 1 & -2 & 2 & 0 & 3 \\ -2 & 5 & -4 & 1 & -5 \\ 2 & -7 & 4 & -2 & 5 \end{pmatrix}$

   (a) Use the Gauss-Jordan method to find the general solution set of $A\boldsymbol{x} = \mathbf{0}$.

   (b) Find the general solution set of the linear system whose augmented matrix is A.

2. (5 points) Determine for what values of k, if any, the following linear system has exactly one solution, no solutions, and infinitely many solutions.
$$\left(\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 1 & k+1 & 2k+1 & 3 \\ -k & -4k & -3 & -9 \end{array}\right)$$

3. Prove the following or give a counterexample, where A and B are $(n \times n)$ matrices.

   (a) (3 points) If $A^4 = 2I$, then the matrix A is invertible.

   (b) (2 points) If $AB = B$ and B is a nonzero matrix, then the matrix A is invertible.

4. (4 points) Write the matrix $A = \begin{pmatrix} 2 & 5 \\ 1 & 2 \end{pmatrix}$ as a product of elementary matrices.

5. (4 points) Let $(B^{-1}A) = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}$ and $BX^{-1}A^{-1} = I_3$. Find the matrix X.

6. (4 points) Use the Cramer's rule to find the value(s) of t for which $x = 1$.
$$\begin{cases} x + 2y + 3z = 2 \\ 2x - y - 2z = 1 \\ x + 3y + tz = -1 \end{cases}$$

7. (4 points) Let A be a (3×3) invertible matrix such that $A^2 = adj(-2A^{-1})$. Find $det(A)$.

8. (4 points) Find all values of t for which the linear system $AX = 0$ has nontrivial solutions, where $A = \begin{pmatrix} 2-t & -1 & 0 \\ 1 & 3+t & -2-t \\ -1 & 0 & 2+t \end{pmatrix}$.

9. (4 points) Find elementary matrices E_1 and E_2 such that $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} = E_1 E_2 \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

10. Given the points $A(1,0,1), B(0,1,1), C(1,1,0), D(1,1,1)$ in $\mathbb{R}^3$.
    (a) (4 points) Find the area of the parallelogram determined by $\overrightarrow{AB}$ and $\overrightarrow{AC}$.
    (b) (4 points) Find the distance from A to the line that passes through B and C.
    (c) (4 points) Determine if the points A, B, C, D lie on the same plane.

11. (4 points) Given $||\vec{u}|| = 1$, $||\vec{v}|| = \sqrt{5}$, $||2\vec{u} - \vec{v}|| = 1$. Find the cosine of the angle between the vectors $\vec{u}$ and $\vec{u} + \vec{v}$.

12. (4 points) Given $Proj_{\vec{v}}(\vec{u} + \vec{v}) = (0, 3, 0)$, where $\vec{v} = (0, 2, 0)$. Find $\vec{u} \cdot \vec{v}$

13. Given the lines L_1: $x = 2+t$, $y = t$, $z = 1+t$ and L_2: $x = 2+s$, $y = -1-s$, $z = 1-s$ $(t, s \in \mathbb{R})$.
    (a) (3 points) Find the general form of the equation of the plane that contains the line L_1 and is parallel to the line L_2.
    (b) (3 points) Find the distance from the plane $x + z = 1$ to the line L_2.

14. (5 points) Find the parametric equations of the line that is perpendicular to the lines $x = 1 + t$, $y = 2 + 2t$, $z = -1 + t$ and $x = 1 - 2s$, $y = 3 - 3s$, $z = 2 + s$, and passes through their point of intersection, where $t, s \in \mathbb{R}$.

15. (4 points) Determine whether $W = \{(x, y, z) : x + y = 3z\}$ is a subspace of $\mathbb{R}^3$.

16. Given the set of all positive real numbers $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$ with the operations $x + y = xy$ and $kx = x^k$.
    (a) (2 points) Does $\mathbb{R}_+$ contain a zero vector? If so find it. Justify.
    (b) (2 points) Does $\mathbb{R}_+$ contain the additive inverse (negative of the vector in the sense of a vector space) for all vectors in $\mathbb{R}_+$? If so find it. Justify.

17. (5 points) Let $\{\vec{u}, \vec{v}, \vec{w}\}$ be a set of linearly independent vectors in $\mathbb{R}^3$. Determine whether the vectors $\vec{u} + 2\vec{v}$, $\vec{v} + 2\vec{w}$, $\vec{u} + \vec{v} + \vec{w}$ are also linearly independent.

18. Given the set $S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 3 & -1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix} \right\}$.
    (a) (3 points) Does the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$ belong to the span of S?
    (b) (3 points) Find a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that if it is added to S, then the set S will form a basis for the vector space of (2×2) matrices $\mathbb{M}_{22}$.

19. (a) (5 points) Show that the set $\{2+x+x^2, 1+2x, 1-x^2\}$ is a basis for $\mathbb{P}_2$.
($\mathbb{P}_2$ is the set of all polynomials of degree 2 or less).

(b) (3 points) Find the coordinates of $1+4x+x^2$ with respect to the basis in part (a).

Answers:

1. $a)(-2t_1 - t_2, t_2, t_1, -2t_2, t_2)$ where $t_1, t_2 \in \mathbb{R}$, b) $(1-2t, -1, t, 2)$ where $t \in \mathbb{R}$.
2. One solution if $k \neq 0, -3$, No solution if $k = 0, -3$, Never
3. a) True, b) False
4. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$
5. $\begin{pmatrix} 0 & -1 & 1 \\ 1 & -1 & 0 \\ -1 & 3 & -1 \end{pmatrix}$
6. $t = 1$
7. $\pm 2\sqrt{2}$
8. $-3, -2, 2$
9. $E_! = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, $E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$
10. a) $\sqrt{3}$, b) $\frac{\sqrt{6}}{2}$, c) No
11. $\frac{3\sqrt{10}}{10}$
12. 2
13. a) $y - z + 1 = 0$, b) $\sqrt{2}$
14. $(3+5t, 6-3t, 1+t)$ where $t \in \mathbb{R}$.
15. Yes
16. a)1, $b)\frac{1}{x}$
17. Yes
18. a)No, b) $\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$
19. b)$(0, 2, -1)$