STUDENT'S NAME: ______________________________________

STUDENT'S ID NUMBER: ____________________________________

INSTRUCTOR'S NAME: ____________________________________

DAWSON COLLEGE – DEPARTMENT OF MATHEMATICS

LINEAR ALGEBRA (201-NYC-05) COMPUTER SCIENCE
FINAL EXAM

MONDAY, DECEMBER 19, 2022 (FROM 09:30 TO 12:30)

INSTRUCTORS: K. AMEUR, Y. LAMONTAGNE, A. JIMENEZ
SECTIONS : 9, 10, 11

NOTE: This exam has 17 questions for a total of 100 marks. No information sheet is provided.
SHOW ALL THE STEPS! NO MARKS ARE GIVEN FOR FINAL ANSWERS ONLY!

INSTRUCTIONS:
Print your name, ID number and instructor's name in the space provided above.
All questions are to be answered in the space provided for each question.
Give detailed solutions and explain your work clearly.
Reverse sides of pages may be used for rough work and/or to complete a solution.
Only calculator permitted is SHARP EL-531***.
This examination consists of 17 questions on 17 pages. Please ensure that you
have a complete examination before starting.

(5 Marks)
Q1:
a) Use the Gauss-Jordan method to find the general solution of the following linear system.

$$\begin{cases} -x_2 - 3x_4 + x_5 = 1 \\ x_1 + x_2 - x_3 = 2 \\ 3x_1 + 3x_2 - x_3 + 2x_4 - 4x_5 = 4 \end{cases}$$

(2 Marks)
b) Find the particular solution for which $x_3 = -1$ and $x_4 = 0$.

Q1

$$-x_2 - 3x_4 + x_5 = 1$$
$$x_1 + x_2 - x_3 = 2$$
$$3x_1 + 3x_2 - x_3 + 2x_4 - 4x_5 = 4$$

$$\left(\begin{array}{ccccc|c} 0 & -1 & 0 & -3 & 1 & 1 \\ 1 & 1 & -1 & 0 & 0 & 2 \\ 3 & 3 & -1 & 2 & -4 & 4 \end{array}\right)$$

$$\left(\begin{array}{ccccc|c} 1 & 1 & -1 & 0 & 0 & 2 \\ 0 & -1 & 0 & -3 & 1 & 1 \\ 0 & 0 & 2 & 2 & -4 & -2 \end{array}\right)$$

$$\left(\begin{array}{ccccc|c} 1 & 1 & -1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 & -1 & -1 \\ 0 & 0 & 1 & 1 & -2 & -1 \end{array}\right)$$

$$\left(\begin{array}{ccccc|c} 1 & 1 & 0 & 1 & -2 & 1 \\ 0 & 1 & 0 & 3 & -1 & -1 \\ 0 & 0 & 1 & 1 & -2 & -1 \end{array}\right)$$

$$\begin{array}{c} \begin{array}{cccccc} & & & t & s & \end{array} \\ \left(\begin{array}{ccccc|c} 1 & 0 & 0 & -2 & -1 & 2 \\ 0 & 1 & 0 & 3 & -1 & -1 \\ 0 & 0 & 1 & 1 & -2 & -1 \end{array}\right) \end{array}$$

$$\begin{cases} x_1 = 2 + s + 2t \\ x_2 = -1 + s - 3t \\ x_3 = -1 + 2s - t \\ x_4 = t \\ x_5 = s \end{cases}$$

$$-1 = -1 + 2s$$

$$\Rightarrow s = 0 = x_5$$

$$\begin{cases} x_1 = 2 \\ x_2 = -1 \\ x_3 = -1 \\ x_4 = 0 \\ x_5 = 0 \end{cases}$$

(6 Marks)

Q2: Determine the value(s) of k for which the following system has

a) no solutions (2 marks)

b) infinitely many solutions (2 marks)

c) exactly one solution (2 marks)

$$\begin{cases} x & +2y & -z & = & 1 \\ 2x & +3y & +z & = & 1 \\ -4x & -5y & +(k^2-9)z & = & k+1 \end{cases}$$

Q2. $\begin{cases} x + 2y - z = 1 \\ 2x + 3y + z = 1 \\ -4x - 5y + (k^2-9)z = k+1 \end{cases}$

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 3 & 1 & 1 \\ -4 & -5 & k^2-9 & k+1 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -1 & 3 & -1 \\ 0 & 3 & k^2-13 & k+5 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & \underline{k^2-4} & \underline{k+2} \end{array}\right)$$

a) $k = 2$ b) $k = -2$

c) $\mathbb{R} \setminus \{-2, 2\}$

(5 Marks)

Q3: Solve for X the following matrix equation.

$$\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} X^{-1} + 3I = \left(X \begin{bmatrix} -1 & 4 \\ 1 & -2 \end{bmatrix} \right)^{-1}$$

Q3 $\begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix} X^{-1} + 3I = \left(X \begin{pmatrix} -1 & 4 \\ 1 & -2 \end{pmatrix} \right)^{-1}$

$$3I = \left(-\frac{1}{2} \begin{pmatrix} -2 & -4 \\ -1 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix} \right) X^{-1}$$

$$3X = \begin{pmatrix} 1-1 & 2-3 \\ \frac{1}{2}-0 & \frac{1}{2}-2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ \frac{1}{2} & -\frac{3}{2} \end{pmatrix}$$

$$X = \frac{1}{3} \begin{pmatrix} 0 & -1 \\ \frac{1}{2} & -\frac{3}{2} \end{pmatrix} = -\frac{1}{3} \begin{pmatrix} 0 & 1 \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix}$$

$$= -\frac{1}{6} \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{3} \\ \frac{1}{6} & -\frac{1}{2} \end{pmatrix}$$

(4 Marks)

Q4: $A=\begin{bmatrix} 0 & 2 & 6 \\ 1 & -4 & -2 \\ 0 & 0 & 1 \end{bmatrix}$. Write A^{-1} as a product of elementary matrices.

Q4 : $\begin{pmatrix} 0 & 2 & 6 \\ 1 & -4 & -2 \\ 0 & 0 & 1 \end{pmatrix}$ $E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\frac{1}{2} \begin{pmatrix} 1 & -4 & -2 \\ 0 & 2 & 6 \\ 0 & 0 & 1 \end{pmatrix}$ $E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\begin{pmatrix} 1 & -4 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} -3R_3$ $E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}$

$\begin{pmatrix} 1 & -4 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{2R_3}$ $E_4 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\begin{pmatrix} 1 & -4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{4R_2}$ $E_5 = \begin{pmatrix} 1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$E_5 E_4 E_3 E_2 E_1 = A^{-1}$

(4 Marks)

Q5: A, B and C are invertible matrices. Simplify the following expression.

$$\left(8A^T AB^2\right)^{-1}\left(4B^{-1}A^T A\right)^T\left(B^T B^3\right)$$

$$(8A^T AB^2)^{-1}(4B^{-1}A^T A)^T \; B^T B^3$$

$$\frac{1}{8} B^{-2}A^{-1}A^{-T} 4 A^T A B^{-T} B^T B^3 = \frac{1}{2} B$$

(3 Marks)

Q6: Assume that $A^T = -A$. Show that if A is symmetric then A=0.

$A^T = A$ and $A^T = -A$

$\Rightarrow A = -A \Rightarrow A = 0$

(3 Marks)

Q7: Determine whether the following statement is true or false. If the statement is true provide a proof of the statement. If the statement is false, provide a counterexample.

If A and B are square matrices and $AB=0$ then A=0 or B=0

$$\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{B} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Q8: B is a 3x3 matrix for which det(B) =3, and $A=\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ where det(A) =-2. Calculate the followings determinants:

(4 Marks)

a) $det\left(adj(3A)(2B)^{-1}(3A^3)^T\right) =$

$$\frac{|(3A)^{-1}|3A||\,|3A^3|}{|2B|}$$

(3 Marks)

b) $det\begin{bmatrix} 3a-d & 4g-2d & -d \\ 3b-e & 4h-2e & -e \\ 3c-f & 4\underset{i}{\cancel{g}}-2\underset{f}{\cancel{e}} & -f \end{bmatrix}$

$$= \frac{|3A|^3\,|3A^3|}{|3A|\,|2B|}$$

$$= \frac{(3^3(-2))^2\,3^3(-2)^3}{2^3\cdot 3} = -4\cdot 3^8$$

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b) $-\begin{vmatrix} 3a & 4g & d \\ 3b & 4h & e \\ 3c & 4i & f \end{vmatrix} = -12\begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix}$

$= -24$

Q9:

(4 Marks)

a) Use the adjoint matrix method to find the inverse of $A=\begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 3 \\ -1 & 1 & -2 \end{bmatrix}$

(2 Marks)

b) Use the result you obtained in part a) to solve the following linear system.

$$\begin{cases} x & -y & z & = & 3 \\ 2x & -y & +3z & = & -2 \\ -x & +y & -2z & = & 1 \end{cases}$$

(2 Marks)

c) Use the Cramer's rule to solve the linear system in b) for y.

$$|A| = \begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ -1 & 0 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ -1 & -2 \end{vmatrix} = -1$$

$$adj(A) = \begin{pmatrix} -1 & 1 & 1 \\ -1 & -1 & 0 \\ -2 & -1 & 1 \end{pmatrix}^T = \begin{pmatrix} -1 & -1 & -2 \\ 1 & -1 & -1 \\ 1 & 0 & 1 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 1 & 1 & 2 \\ -1 & 1 & 1 \\ -1 & 0 & -1 \end{pmatrix}$$

b)

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ -1 & 1 & 1 \\ -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3-2+2 \\ -3-2+1 \\ -3-1 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ -4 \end{pmatrix}$$

c) $$\begin{vmatrix} 1 & 3 & 1 \\ 2 & -2 & 3 \\ -1 & 1 & -2 \end{vmatrix}_y = \begin{vmatrix} 1 & 4 & 1 \\ 2 & 0 & 3 \\ -1 & 0 & -2 \end{vmatrix} = -4(-4+3) = 4$$

$$\Rightarrow y = \frac{4}{-1} = -4$$

(4 Marks)

Q10: Evaluate the following determinant.

$$\begin{vmatrix} 3 & -3 & 4 & 3 \\ -2 & 4 & 5 & 0 \\ 5 & -9 & 10 & 9 \\ 6 & -1 & 8 & 3 \end{vmatrix} = \begin{vmatrix} 3 & -3 & 4 & 3 \\ -2 & 4 & 5 & 0 \\ -4 & 0 & -2 & 0 \\ 3 & 2 & 4 & 0 \end{vmatrix}$$

$$-3 \begin{vmatrix} -2 & 4 & 5 \\ -4 & 0 & -2 \\ 3 & 2 & 4 \end{vmatrix} = -3 \begin{vmatrix} -12 & 4 & 5 \\ 0 & 0 & -2 \\ -5 & 2 & 4 \end{vmatrix}$$

$$= -3 \cdot 2 \left(-24+20\right) = 24$$

Q11: Let $\vec{u}=(1,1,-1)$, $\vec{v}=(2,1,-1)$ and $\vec{w}=(0,-1,1)$. Answer the following questions.

(3 Marks)

a) A unit vector in the opposite direction of $\vec{v}$.

(3 Marks)

b) The angle between $\vec{u}$ and $\vec{v}+\vec{w}$. $= (2,0,0)$

(3 Marks)

c) Determine whether vectors $\vec{u},\vec{v},\vec{w}$ lie on the same plane.

(3 Marks)

d) An equation of the plane passing through P(1,0,1) and parallel to $\vec{u}$ and $\vec{v}$.

a) $$\frac{1}{\sqrt{2^2+1+1}}(-2,-1,1) = \left(-\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

b) $$\arccos \frac{(1,1,-1)(2,0,0)}{2\sqrt{3}}$$

$$= \arccos\left(\frac{1}{\sqrt{3}}\right) \approx 54,735^\circ$$

c) $$\begin{vmatrix} 1 & 1 & -1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -1 \\ 2 & 0 & -1 \\ 0 & 0 & 1 \end{vmatrix} = 0$$

d) $\bar{u} = (1, 1, -1)$ $\bar{v} = (2, 1, -1)$

$P(1, 0, 1)$

$$(x, y, z) = (1, 0, 1) + t(1, 1, -1) + s(2, 1, -1)$$

or point normal version

Q12: Given points A(6,-3,-2), B(1,1,3) and C(2,-1,2) answer the following questions.

(4 Marks)

a) Find the area of the triangle ABC.

(4 Marks)

b) Find the point on the line passing through B and C which is closest to A.

(4 Marks)

c) Find the distance between A and the line passing through B and C.

a) $\vec{BA} = (5,-4,-5)$

$\vec{BC} = (1,-2,-1)$

$\vec{BA} \times \vec{BC} = (-6, 0, -6)$

$A_\Delta = \frac{1}{2} \| \vec{BA} \times \vec{BC} \| = \frac{1}{2} \sqrt{6^2+6^2} =$

$\frac{6}{2}\sqrt{2} = 3\sqrt{2}$

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b) $\bar{v} = \vec{BC} = (1,-2,-1)$

$L: (x,y,z) = (1+t, 1-2t, 3-t)$

Closest point P to A on L

$P(1+t, 1-2t, 3-t)$

$A(6,-3,-2)$

So $\vec{AP} \cdot \vec{v} = 0$

$$(-5+t,\ 4-2t,\ 5-t)(1,-2,-1)=0$$

$$-5+t-8+4t-5+t=0$$

$$6t=18 \Rightarrow t=3 \Rightarrow P(4,-5,0)$$

c) $d = \|\vec{AP}\| = \sqrt{2^2+2^2+2^2} = 2\sqrt{3}$

(4 Marks)

Q13: Find an equation of the line passing through point A(1,-1,4) and is parallel to the line of intersection of plane P1 and plane P2.

P1: x-2y+2z=2

P2: 3x-4y+8z=0

$$\left(\begin{array}{ccc|c} 1 & -2 & 2 & 2 \\ 3 & -4 & 8 & 0 \end{array}\right) \to \begin{array}{ccc|c} 1 & -2 & 2 & 2 \\ 0 & 2 & 2 & -6 \end{array}$$

$$\to \begin{cases} z = t \\ y = -3 - t \\ x = 2 - 2t + 2(-3-t) = -4 - 4t \end{cases}$$

$$\Rightarrow \vec{v} = (-4, -1, 1)$$

$$L: (x, y, z) = (1, -1, 4) + t(-4, -1, 1)$$

(4 Marks)

Q14: Determine whether L1 and L2 are skew lines. If they are skew lines calculate the distance between them. If they are not skew lines write the equation of the plane passing through L1 and L2.

L1: (x, y, z) = (2+t, 1-t, 4+t)

L2: (x, y, z) = (s, -1+s, 2-s)

$P_1(2,1,4)$ on L_1

$P_2(1,-1,2)$ on L_2

$\vec{v}_1 = (1,-1,1)$

$\vec{v}_2 = (1,1,-1)$

$\vec{v}_1 \nparallel \vec{v}_2 \quad \left(\frac{1}{1} \neq \frac{-1}{1}\right)$

$$d = \| proj_{\vec{v}_1 \times \vec{v}_2} \overline{P_1P_2} \| = \frac{|(-1,-2,-2)(0,2,2)|}{2\sqrt{2}}$$

$$= \frac{8}{2\sqrt{2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2} \neq 0$$

$\Rightarrow L_1$ and L_2 are skew - lines

Q15: Given plane P1: x-3y+2z–4 = 0 and plane P2: -3x+9y-6z-6=0 answer the following questions.

(3 Marks)

a) Show that P1 and P2 are parallel and find the distance between them.

(4 Marks)

b) Find the point on plane P1 closest to point A(2,8,-1).

a) $\bar{n}_1 = (1, -3, 2)$ normal to p_1

$\bar{n}_2 = (-3, 9, -6)$ normal to p_2

$$\frac{1}{-3} = \frac{-3}{9} = \frac{2}{-6} = -\frac{1}{3} \Rightarrow \vec{n}_1 \parallel \vec{n}_2$$

$\Rightarrow P_1 \parallel P_2$

$P_2(-2, 0, 0)$ on P_2

$$d(P_1, P_2) = d(P_2, P_1) = \frac{|-2-4|}{\sqrt{1+3^2+2^2}}$$

$$= \frac{6}{\sqrt{14}}$$

b) $p_1: \ x - 3y + 2z - 4 = 0$

$P(2, 8, -1)$

$L: (x, y, z) = (2, 8, -1) + (1, -3, 2)t$

$= (2 + t, 8 - 3t, -1 + 2t)$

$2 + t - 3(8 - 3t) + 2(-1 + 2t) - 4 = 0$

$14t + 2 - 24 - 2 - 4 = 0$

$\Rightarrow t = 2 \quad \Rightarrow$ closest point

$C(4, 2, 3)$

(5 Marks)

Q16: Maximize $\quad P = 5\,x_1 + 2\,x_2 + x_3$

subject to
$$\begin{cases} x_1 + 3x_2 - x_3 \le 6 \\ x_2 + x_3 \le 4 \\ 3x_1 + x_2 \le 7 \end{cases}$$
$$x_1, x_2, x_3 \ge 0$$

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 6 \\ 0 & 1 & 1 & 4 \\ \boxed{3} & 1 & 0 & 7 \\ \hline -5 & -2 & -1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 3 & -1 & 6 \\ 0 & 1 & 1 & 4 \\ \boxed{1} & 1/3 & 0 & 7/3 \\ \hline -5 & -2 & -1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 0 & 8/3 & -1 & 11/3 \\ 0 & 1 & \boxed{1} & 4 \\ 1 & 1/3 & 0 & 7/3 \\ \hline 0 & -\frac{1}{3} & -1 & 35/3 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 0 & 11/3 & 0 & 23/3 \\ 0 & 1 & 1 & 4 = x_3 \\ 1 & 1/3 & 0 & 7/3 = x_1 \\ \hline 0 & 2/3 & 0 & 47/3 \end{array}\right]$$

Max

$x_2 = 0$

$$\boxed{P = 5 \cdot \frac{7}{3} + 4 = \frac{35}{3} + \frac{12}{3} = \frac{47}{3}}$$

(5 Marks)

Q17: Minimize $C = x_1 + x_2 + 4x_3$

subject to
$$\begin{cases} -x_1 - x_2 + x_3 \geq 2 \\ -x_1 + x_2 + 2x_3 \geq 1 \end{cases}$$
$$x_1, x_2, x_3 \geq 0$$

$$\begin{pmatrix} -1 & -1 & 1 & 2 \\ -1 & 1 & 2 & 1 \\ 1 & 1 & 4 & 0 \end{pmatrix}^T = \begin{pmatrix} -1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 2 & 4 \\ 2 & 1 & 0 \end{pmatrix}$$

$$\left(\begin{array}{cc|ccc|c} -1 & -1 & 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 1 & 0 & 1 \\ \boxed{1} & 2 & 0 & 0 & 1 & 4 \\ \hline -2 & -1 & 0 & 0 & 0 & 0 \end{array}\right) \rightarrow$$

$$\begin{array}{cc|ccc|cl}
0 & 1 & 1 & 0 & 1 & 5 & \\
0 & 3 & 0 & 1 & 1 & 5 & \\
1 & 2 & 0 & 0 & 1 & 4 & \\
\hline
0 & 3 & 0 & 0 & 2 & 8 & \text{min} \\
 & & & & \shortparallel & & \\
 & & & & x_3 & &
\end{array}$$

$$P = 2 \cdot 4 = 8$$