DAWSON COLLEGE
MATHEMATICS DEPARTMENT
Linear Algebra (SCIENCE)

201-NYC-05
Winter 2019
Final Examination
May 24th, 2019
Time Limit: 3 hours

Name: ______________________________

ID#: ______________________________

Instructor: A. Douba, A. Gambioli, Y. Lamontagne, M. Moodi, S. Muise, B. Szczepara.

- This exam contains 16 pages (including this cover page) and 18 problems. Check to see if any pages are missing.
- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page, and please indicate that you have done so.
- Give the work in full; – unless otherwise stated, reduce each answer to its simplest, exact form; – and write and arrange your exercise in a legible and orderly manner.
- You are only permitted to use the **Sharp EL-531X, XG** or **XT** calculator.
- This examination booklet must be returned intact.
- Good luck!

Question	Points	Score
1	6	
2	6	
3	5	
4	4	
5	4	
6	4	
7	4	
8	5	
9	10	
10	5	
11	8	
12	5	
13	5	
14	7	
15	5	
16	5	
17	6	
18	6	
Total:	100	

Problem 1.(6 marks)

a)(4 marks) Solve the linear system using Gauss-Jordan:

$$\begin{cases} 2x_1 + 2x_2 + 2x_3 + 4x_4 = 6 \\ 2x_1 + 3x_2 + 3x_3 + 7x_4 = 10 \\ 3x_1 + 3x_2 + 4x_3 + 7x_4 = 10 \end{cases}$$

$$\begin{bmatrix} 2 & 2 & 2 & 4 & 6 \\ 2 & 3 & 3 & 7 & 10 \\ 3 & 3 & 4 & 7 & 10 \end{bmatrix} \xrightarrow{R_1/2} \begin{bmatrix} 1 & 1 & 1 & 2 & 3 \\ 2 & 3 & 3 & 7 & 10 \\ 3 & 3 & 4 & 7 & 10 \end{bmatrix} \xrightarrow[R_2-2R_1]{} \begin{bmatrix} 1 & 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 3 & 4 \\ 3 & 3 & 4 & 7 & 10 \end{bmatrix} \to$$

$$\xrightarrow[R_3-3R_1]{} \begin{bmatrix} 1 & 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 3 & 4 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \xrightarrow[R_1-R_2]{} \begin{bmatrix} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & 1 & 3 & 4 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \xrightarrow[R_2-R_3]{} \begin{bmatrix} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 2 & 3 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

RREF

$$\boxed{\begin{aligned} x_1 &= -1+t \\ x_2 &= 3-2t \\ x_3 &= 1-t \\ x_4 &= t \end{aligned}}$$

$(x_1 x_2 x_3 x_4) = (-1+t, 3-2t, 1-t, t)$

b)(2 marks) Solve the homogeneous system whose coefficient matrix is the augmented matrix of the system in part a).

$$\begin{bmatrix} 2 & 2 & 2 & 4 & 6 & 0 \\ 2 & 3 & 3 & 7 & 10 & 0 \\ 3 & 3 & 4 & 7 & 10 & 0 \end{bmatrix} \xrightarrow[\text{SAME PROCESS AS IN (a)}]{} \cdots \begin{bmatrix} 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 2 & 3 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

$$\boxed{\begin{aligned} x_1 &= s+t \\ x_2 &= -2s-3t \\ x_3 &= -s-t \\ x_4 &= s \\ x_5 &= t \end{aligned}}$$

$(x_1 x_2 x_3 x_4 x_5) = (s+t, -2-3t, -s-t, s, t)$

Problem 2.(6 marks) Given the matrix

$$A = \begin{pmatrix} 5 & 3 & 0 \\ 3 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

a)(3 marks) find A^{-1}.

$$\left[\begin{array}{ccc|ccc} 5 & 3 & 0 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1/5} \left[\begin{array}{ccc|ccc} 1 & 3/5 & 0 & \frac{1}{5} & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2-3R_1} \left[\begin{array}{ccc|ccc} 1 & 3/5 & 0 & \frac{1}{5} & 0 & 0 \\ 0 & \frac{1}{5} & 1 & -3/5 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{5R_2} \left[\begin{array}{ccc|ccc} 1 & 3/5 & 0 & \frac{1}{5} & 0 & 0 \\ 0 & 1 & 5 & -3 & 5 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1-\frac{3}{5}R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 2 & -3 & 0 \\ 0 & 1 & 5 & -3 & 5 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow$$

$$\xrightarrow{R_2-5R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 2 & -3 & 0 \\ 0 & 1 & 0 & -3 & 5 & -5 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1+3R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -3 & 3 \\ 0 & 1 & 0 & -3 & 5 & -5 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{pmatrix} 2 & -3 & 3 \\ -3 & 5 & -5 \\ 0 & 0 & 1 \end{pmatrix} = A^{-1}$$

b)(3 marks) Write A as a product of elementary matrices.

Using The same operations used in (a)

$$E_6 E_5 E_4 E_3 E_2 E_1 A = I$$

$$\rightarrow A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1} E_6^{-1} =$$

$$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ +3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{3}{5} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Problem 3.(5 marks) Solve for A the following equation:

$$(3A^T - I)^{-1}C - D = 0$$

where $C = \begin{pmatrix} 7 & -1 \\ 6 & -1 \end{pmatrix}$ and $D = \begin{pmatrix} 5 & 4 \\ 1 & 1 \end{pmatrix}$.

$$D^{-1} = \begin{bmatrix} 1 & -4 \\ -1 & 5 \end{bmatrix}$$

$$(3A^T - I)^{-1} = DC^{-1} \rightarrow 3A^T - I = (DC^{-1})^{-1} = CD^{-1}$$

$$= \begin{bmatrix} 7 & -1 \\ 6 & -1 \end{bmatrix}\begin{bmatrix} 1 & -4 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 8 & -33 \\ 7 & -29 \end{bmatrix}$$

$$\rightarrow 3A^T = I + \begin{bmatrix} 8 & -33 \\ 7 & -29 \end{bmatrix} = \begin{bmatrix} 9 & -33 \\ 7 & -28 \end{bmatrix}$$

$$\rightarrow A^T = \frac{1}{3}\begin{bmatrix} 9 & -33 \\ 7 & -28 \end{bmatrix} = \begin{bmatrix} 3 & -11 \\ \frac{7}{3} & -\frac{28}{3} \end{bmatrix}$$

$$\rightarrow \boxed{A = \begin{bmatrix} 3 & 7/3 \\ -11 & -\frac{28}{3} \end{bmatrix}}$$

Problem 4.(4 marks) A matrix A is called *nilpotent* if $A^k = 0$ for some positive integer k. Show that if A is nilpotent for a certain k then

$$(I - A)^{-1} = I + A + A^2 + \cdots + A^k$$

$$(I-A)(I+A+A^2 - + A^k) = I + \cancel{A} + \cancel{A^2} + - + \overset{0}{\overset{=}{A^k}} +$$

$$+ \cancel{(-A)} + \cancel{(-A^2)} + - + (-A^{k+1})$$

$$= I - \underset{\underset{\underset{0}{\shortparallel}}{\underset{-A A^k}{\shortparallel}}}{A^{k+1}} = I \quad \checkmark$$

Problem 5.(4 marks) Show that if A and B are square matrices such that AB is invertible, then A can be written as a product of elementary matrices.

IF AB IS INVERTIBLE → EACH FACTOR IS INVERTIBLE (BOTH A AND B) (THEOREM 1.6.5)

→ BY EQUIVALENCE THEOREM: A INVERTIBLE ⇔ A CAN BE WRITTEN AS PRODUCT OF ELEMENTARY MATRICES.

Problem 6.(4 marks) Solve the linear system for x and y using Cramer's rule:

$$\begin{cases} 17x + 23y = 63 \\ -13x + 31y = 49 \end{cases}$$

$$|A| = \begin{vmatrix} 17 & 23 \\ -13 & 31 \end{vmatrix} = 527 + 299 = 826 \neq 0$$

$$|A_1| = \begin{vmatrix} 63 & 23 \\ 49 & 31 \end{vmatrix} = 1953 - 1127 = 826$$

$$|A_2| = \begin{vmatrix} 17 & 63 \\ -13 & 49 \end{vmatrix} = 833 + 819 = 1652$$

$$\rightarrow x_0 = \frac{826}{826} = 1 \; ; \; y_0 = \frac{1652}{826} = 2$$

$$\boxed{(x_0, y_0) = (1, 2)}$$

Problem 7.(4 marks) Let A and B be two 3×3 matrices with $\det(A) = 2$ and $\det(B) = 5$. Find:

$$\det(A^3 \det(B) A^T \,(\mathrm{adj}(A))^4).$$

$$\mathrm{adj}(A) = \det(A)\, A^{-1} = 2A^{-1}$$

$$\det(A^3 \underbrace{\det(B)}_{=5} A^T (\mathrm{adj}(A))^4) = 5^3 \det(A)^3 \underbrace{\det(A^T)}_{=\det(A)} (\det(2A^{-1}))^4$$

$$= 5^3 \cancel{2^3} \cdot \cancel{2} \quad 2^{12}\, \cancel{2}^{-4}$$

$$= 5^3 \cdot 2^{12} = 512000$$

Problem 8.(5 marks) Find the determinant of A:

$$A = \begin{pmatrix} 3 & 2 & -1 & 4 \\ 2 & 1 & 5 & 7 \\ 0 & 5 & 2 & -6 \\ -1 & 2 & 1 & 0 \end{pmatrix}.$$

$$\det(A) \underset{R_1 \leftrightarrow R_4}{=} - \begin{vmatrix} -1 & 2 & 1 & 0 \\ 2 & 1 & 5 & 7 \\ 0 & 5 & 2 & -6 \\ 3 & 2 & -1 & 4 \end{vmatrix} = + \begin{vmatrix} 1 & -2 & 1 & 0 \\ 2 & 1 & 5 & 7 \\ 0 & 5 & 2 & -6 \\ 3 & 2 & -1 & 4 \end{vmatrix} \underset{R_2 \leftrightarrow R_3}{=}$$

$$= - \begin{vmatrix} 1 & -2 & 1 & 0 \\ 0 & 5 & 2 & -6 \\ 2 & 1 & 5 & 7 \\ 3 & 2 & -1 & 4 \end{vmatrix} \underset{\substack{R_3 - 2R_1 \\ R_4 - 3R_1}}{=} - \begin{vmatrix} 1 & -2 & -1 & 0 \\ 0 & 5 & 2 & -6 \\ 0 & 5 & 7 & 7 \\ 0 & 8 & 2 & 4 \end{vmatrix} =$$

$$\underset{R_3 - R_2}{=} - \begin{vmatrix} 1 & -2 & -1 & 0 \\ 0 & 5 & 2 & -6 \\ 0 & 0 & 5 & 13 \\ 0 & 8 & 2 & 4 \end{vmatrix} = -\left(5 \begin{vmatrix} 5 & 13 \\ 2 & 4 \end{vmatrix} + 8 \begin{vmatrix} 2 & -6 \\ 5 & 13 \end{vmatrix}\right)$$

$$= -\left(5(20 - 26) + 8(26 + 30)\right)$$

$$= -(-30 + 448) = -418$$

Problem 9.(10 marks)

a)(5 marks) Find the closest point to the point $P = (1, 1, -2)$ on the plane Π of equation $2x - 3y + z = 1$.

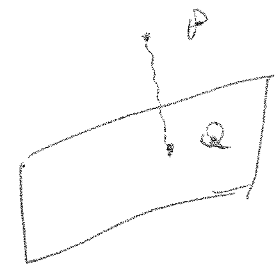
$L: (1,1,-2) + t(2,-3,1) = (1+2t, 1-3t, -2+t)$

$n = (2,-3,1)$

$L \cap \Pi: 2(1+2t) - 3(1-3t) + (-2+t) = 1$

$\rightsquigarrow \cancel{2} + 4t - 3 + 9t \cancel{-2} + t = 1$

$\rightsquigarrow 14t = 4 \rightsquigarrow t = \frac{4}{14} = \frac{2}{7}$



$Q = \left(1 + \frac{4}{7}, 1 - \frac{6}{7}, -2 + \frac{2}{7}\right) = \boxed{\left(\frac{11}{7}, \frac{1}{7}, \frac{-12}{7}\right)}.$

b)(5 marks) Find the distance between the lines

$$L_1 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}_{P} + r \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad L_2 : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{Q} + t \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

$$n = (1,1,1) \times (0,1,-1) = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 0 & 1 & -1 \end{vmatrix} = \left(\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}, -\begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \right)$$

$= (-2, 1, 1).$

$\vec{PQ} = (0,0,1)$

$$d = \|\mathrm{Proj}_n \vec{PQ}\| = \frac{|(0,0,1) \cdot (-2,1,1)|}{\sqrt{4+1+1}} = \frac{1}{\sqrt{6}} = \frac{\sqrt{6}}{6}$$

Problem 10.(5 marks) Let Π be the plane in $\mathbb{R}^3$ with normal vector

$$\mathbf{n} = \begin{pmatrix} \cos\theta \\ \cos\theta + \sin\theta \\ \sin\theta \end{pmatrix}.$$

Find all the values of θ in the interval $[0, 2\pi)$ such that Π is parallel to the plane of vector equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + r \underbrace{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}}_{= v} + t \underbrace{\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}}_{= w}$$

.

$$n_2 = v \times w = \begin{vmatrix} i & j & k \\ 1 & -1 & 1 \\ 1 & -1 & -2 \end{vmatrix} = \left(\begin{vmatrix} -1 & 1 \\ -1 & -2 \end{vmatrix}, -\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix}, \begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix} \right)$$

$$= (3, 3, 0)$$

The plane Π is parallel to the other plane iff $(\cos\theta, \cos\theta + \sin\theta, \sin\theta) = h(3, 3, 0)$ for some h

$\to h = \frac{1}{3} \to \begin{cases} \cos\theta = 1 \\ \cos\theta + \sin\theta = 1 \\ \sin\theta = 0 \end{cases} \to \theta = 0 + 2k\pi,\ k \text{ in } \mathbb{Z}$

$\to \boxed{\theta = 0}$ in $[0, 2\pi)$.

$h = -\frac{1}{3} \to \begin{cases} \cos\theta = -1 \\ \cos\theta + \sin\theta = -1 \\ \sin\theta = 0 \end{cases} \to \boxed{\theta = \pi}$ in $[0, 2\pi)$

Problem 11.(8 marks) Given

$$\mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 3 \\ 4 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{w} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ -1 \end{pmatrix}$$

in $\mathbb{R}^4$ evaluate the following (if they exist):

a) $\mathrm{proj}_{\mathbf{u}}(\mathbf{v}+\mathbf{w})$ $\quad V + W = (3,5,3,-1)$

$$\mathrm{Proj}_U(V+W) = \frac{U \cdot (V+W)}{\|U\|^2} U = \frac{3+5-2}{1+1+4}(1,1,0,2)$$

$$= \frac{6}{6}(1,1,0,2) = (1,1,0,2)$$

b) $\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + \mathbf{v}\cdot\mathbf{w}$

$$= (1+1+4) + (9+16+1) + (0+4+2+0)$$

$$= 6 + 26 + 6 = 38$$

c) $\mathbf{u} \times \mathbf{v} - \mathbf{u} \cdot \mathbf{w}$

VECTOR SCALAR

The expression does not make sense!

(MOREOVER: $u \times v$ IS DEFINED only in $\mathbb{R}^3$ while u, v ARE IN $\mathbb{R}^4$!)

d) the angle θ between $\mathbf{u}$ and $\mathbf{v}$

$$\theta = \arccos\left(\frac{u \cdot v}{\|u\| \|v\|}\right) = \arccos\left(\frac{3+4+0+0}{\sqrt{6}\,\sqrt{26}}\right)$$

$$= \arccos\left(\frac{7}{\sqrt{6}\sqrt{26}}\right) \simeq 55.91^\circ$$

Problem 12.(5 marks) A parallelepiped has the vectors

$$\mathbf{u} = \begin{pmatrix} a \\ 2 \\ 1 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 0 \\ a \\ -2 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

as its edges. Find the values of a so that it will have volume $V = 8$.

$$8 = \text{Volume} = \begin{vmatrix} a & 2 & 1 \\ 0 & a & -2 \\ 1 & 1 & 1 \end{vmatrix} = a \begin{vmatrix} a & -2 \\ 1 & 1 \end{vmatrix} + 1 \begin{vmatrix} 2 & 1 \\ a & -2 \end{vmatrix} = a(a+2) + (-4-a)$$

$$= a^2 + 2a - 4 - a = a^2 + a - 4$$

$$\rightarrow a^2 + a - 4 = 8 \rightarrow a^2 + a - 12 = 0$$

$$\rightarrow a = \frac{-1 \pm \sqrt{1+48}}{2} = \frac{-1 \pm \sqrt{49}}{2} = \frac{-1 \pm 7}{2} = -4, \; 3$$

Problem 13.(5 marks) Show that if $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are vectors in a vector space V, then $\mathrm{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\} = \mathrm{Span}\{\mathbf{u}, \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{w}\}$.

$\supseteq$: $u, u+v, u+w$ ARE LINEAR COMBINATIONS OF u, v, w ✓

$\subseteq$: $v = au + b(u+v) + c(u+w) \rightarrow a=-1, b=1, c=0$

$w = au + b(u+v) + c(u+w) \rightarrow a=-1, b=0, c=1$ ✓

Problem 14.(7 marks) Consider the vector space $S_{2\times 2}$ of symmetric matrices of size 2×2.

a)(5 marks) Show that the vectors

$$A_1 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \quad A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad A_3 = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

form a basis for $S_{2,2}$.

$$aA_1 + bA_2 + cA_3 = \begin{cases} 0 & \text{L. INDEP.} \\ \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} & \text{span } S_{2\times 2} \end{cases}$$

$$\rightarrow \begin{cases} 2a+b=0 \\ 2c=0 \\ a+2b=0 \end{cases} \quad \text{(L. indep)} \quad \text{OR} \quad \begin{cases} 2a+b=a_{11} \\ 2c=a_{12} \\ a+2b=a_{22} \end{cases}$$

$$\downarrow$$

$$\begin{vmatrix} 2 & 1 & 0 \\ 0 & 0 & 2 \\ 1 & 2 & 0 \end{vmatrix} = -2\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = -2(4-1) = -6 \neq 0 \rightarrow \text{YES, IT'S A BASIS.}$$

b)(2 mark) Write the components of the vector B with respect to the basis given in a).

$$B = \begin{pmatrix} 5 & -9 \\ -9 & 7 \end{pmatrix}$$

$$\begin{cases} 2a+b=5 \\ 2c=-9 \\ a+2b=7 \end{cases} \rightarrow$$

$c = -\frac{9}{2}$

$3a+3b=12 \rightarrow a+b=4$

$a = 4-b$

$8-2b+b=5 \rightarrow b=3$

$a=1$

$$\rightarrow \boxed{(B)_S = \left(1, 3, -\tfrac{9}{2}\right)}$$

$S = \{A_1 A_2 A_3\}$

Problem 15.(5 marks) Consider the vectors in $\mathbb{R}^3$

$$\mathbf{u} = \begin{pmatrix} k \\ 1 \\ 0 \end{pmatrix} \quad , \quad \mathbf{v} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \quad , \quad \mathbf{w} = \begin{pmatrix} 3 \\ 1 \\ k \end{pmatrix} .$$

Find the values of k that make $S = \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ a basis for $\mathbb{R}^3$.

S BASIS $\Leftrightarrow \det \begin{bmatrix} k & 2 & 3 \\ 1 & 1 & 1 \\ 0 & 1 & k \end{bmatrix} \neq 0$; $\det \begin{bmatrix} k & 2 & 3 \\ 1 & 1 & 1 \\ 0 & 1 & k \end{bmatrix} =$

$= k \begin{vmatrix} 1 & 1 \\ 1 & k \end{vmatrix} - 1 \begin{vmatrix} 2 & 3 \\ 1 & k \end{vmatrix} = k(k-1) - (2k-3) = k^2 - k - 2k + 3 =$

$= k^2 - 3k + 3 = 0 \Leftrightarrow k = \dfrac{3 \pm \sqrt{9-12}}{2} \rightarrow \Delta < 0$ No Solution

S BASIS FOR ANY k IN $\mathbb{R}$.

Problem 16.(5 marks) Consider the vector space $P_4(x)$ of polynomials of degree ≤ 4 and the subset of $P_4(x)$ defined by

$$W_k = \{p(x) = a_4 x^4 + \cdots + a_0 \text{ such that } p(1) = k \text{ for some integer } k\} .$$

Find the values of k for which W_k is a vector subspace of $P_4(x)$ (if any).

Axiom 1 : $P_1(x) + P_2(x) \rightarrow P_1(1) + P_2(1) = k + k = 2k$

$\rightarrow 2k = k \Leftrightarrow k = 0$

Axiom 6 : $\ell P_1(1) = \ell k = k$ FOR any $\ell \in \mathbb{R}$

$\Leftrightarrow (\ell - 1) k = 0 \Leftrightarrow k = 0$

$\boxed{\text{ONLY FOR } k = 0}$

Problem 17.(6 marks)

a)(3 marks) Find the conditions on b_1, b_2 and b_3 such that the system:

$$\begin{cases} x + 2y + 3z + w = b_1 \\ 3x + 5y - 4z - w = b_2 \\ 4x + 7y - z = b_3 \end{cases}$$

is consistent.

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 1 & b_1 \\ 3 & 5 & -4 & -1 & b_2 \\ 4 & 7 & -1 & 0 & b_3 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 3 & 1 & b_1 \\ 0 & -1 & -13 & -4 & b_2-3b_1 \\ 0 & -1 & -13 & -4 & b_3-4b_1 \end{array}\right] \rightarrow$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 3 & 1 & b_1 \\ 0 & 1 & 13 & 4 & 3b_1-b_2 \\ 0 & 0 & 0 & 0 & -b_1+b_3-b_2 \end{array}\right] \rightarrow \boxed{b_1+b_2-b_3=0}$$

TO BE CONSISTENT

b)(3 marks) Is the collection of all the $b = (b_1, b_2, b_3)$ that satisfy the condition found in part a) a subspace of $\mathbb{R}^3$? If it is, find its dimension and a basis for it.

YES: $b_1+b_2-b_3=0$ IS A 2-plane through the origin

$\rightarrow$ SUBSPACE OF $\mathbb{R}^3$.

$[1 \ 1 \ -1 \ 0] \rightarrow (b_1, b_2, b_3) = (-s+t, s, t)$

$b_2 = s$

$b_3 = t$

$$= s\underbrace{(-1,1,0)}_{v} + t\underbrace{(1,0,1)}_{w}$$

$S = \{v, w\}$ IS LIN. INDEP.

$\rightarrow$ BASIS $\rightarrow \dim(\text{Span}(S)) = 2$

Problem 18.(6 marks) Determine whether the following statements are true or false, providing an appropriate proof or counter-example.

a)(3 marks) For two square matrices A and B of the same size, $\det(A+B) = \det(A) + \det(B)$.

FALSE! EX: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $A+B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$\rightarrow \det(A) = \det(B) = 1$

But $\det(A+B) = 4 \neq 1+1 = 2$.

b)(3 marks) If W is a subset of $\mathbb{R}^3$ that contains a line through the origin, then it is a vector subspace of $\mathbb{R}^3$.

FALSE. EX:

$W = \{(x,0,0) \cup (0,y,0)$ for all x, y in $\mathbb{R}\}$

contains 2 lines, but

$(x,0,0) + (0,y,0) = (x,y,0)$ is NOT in W (Axiom 1 fails).