DAWSON COLLEGE
MATHEMATICS DEPARTMENT
FINAL EXAMINATION
201-NYC-05: Linear Algebra (SCIENCE)
Winter 2022

1. Given the linear system

$$\begin{cases} x_1 + 2x_2 + x_3 + 3x_4 = 4 \\ 2x_1 + 3x_2 + 3x_3 + 4x_4 = 7 \\ -x_1 + x_2 - 4x_3 + 3x_4 = -1 \\ 3x_1 + 5x_2 + 4x_3 + 7x_4 = 11 \end{cases}$$

(a) (5 marks) Solve using Gauss-Jordan elimination.

Answer: $(x_1,\ x_2,\ x_3,\ x_4) = (2-3s+t,\ 1+s-2t,\ s,\ t)$ where $s,t \in \mathbb{R}$

(b) (2 marks) Find a particular solution for which $x_1 = -4$, and $x_2 = 8$.

Answer: $(x_1,\ x_2,\ x_3,\ x_4) = (-4,\ 8,\ 1,\ -3)$

2. Let $A = \begin{bmatrix} 0 & 1 & -3 \\ -2 & 0 & 0 \\ 5 & 0 & 1 \end{bmatrix}$

(a) (3 marks) Find A^{-1}.

Answer: $A^{-1} = \begin{bmatrix} 0 & -1/2 & 0 \\ 1 & 15/2 & 3 \\ 0 & 5/2 & 1 \end{bmatrix}$

(b) (4 marks) Express A as a product of elementary matrices.

Answer: $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5/2 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$

3. Given $A = \begin{bmatrix} 0 & e^t & 0 \\ e^t \sin t & 0 & -e^t \cos t \\ e^{-t} \cos t & 0 & e^{-t} \sin t \end{bmatrix}$.

(a) (3 marks) Determine the value(s) of t for which A is invertible.

Answer: $t \in \mathbb{R}$

(b) (1 mark) Find $A^2\text{adj}(A^2)$ without computing $\text{adj}(A^2)$.

Answer: $\begin{bmatrix} e^{2t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix}$

4. (5 marks) Solve for X where $\left(\frac{1}{2}A + B(X^{-1})^T\right)^{-1} = X^T A^{-1}$, $A = \begin{bmatrix} 1 & 4 \\ 1 & 6 \end{bmatrix}$, and $B = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$.

Answer: $X = \begin{bmatrix} -4 & 1 \\ 2 & 1 \end{bmatrix}$

5. (4 marks) Let A be an $n \times m$ matrix, such that $A^T A = I_m$. Show that $I_n - 2AA^T$ is its own inverse and symmetric.

Answer: Show that $(I_n - 2AA^T)(I_n - 2AA^T) = I$ and $(I_n - 2AA^T)^T = I_n - 2AA^T$

6. Given $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & -5 \end{bmatrix}$

(a) (1 mark) Compute AB^T.

Answer: $\begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix}$

(b) (3 marks) Find all symmetric matrices X, if any, that satisfy $AB^T(I - X) = 0$.

Answer: $X = \begin{bmatrix} 1+t & t \\ t & 1+t \end{bmatrix}$ where $t \in \mathbb{R}$

7. (5 marks) If $A = \begin{bmatrix} 5 & 8 & 1 \\ 0 & 1 & 9 \\ 0 & 0 & 2 \end{bmatrix}$ compute $\det\left(4\det(A^T)\, A^{-1}\, \text{adj}(A^{-1})\right)$.

Answer: 64

8. (5 marks) Let $P_0(-4,3,7)$ be a point in $\mathbb{R}^3$ and $\mathscr{L}:\ x=2+t,\ y=1-2t,\ z=3+t$ where $t\in\mathbb{R}$ be a line in $\mathbb{R}^3$. Find the equation of the line which passes through the point P_0 and intersects the line $\mathscr{L}$ at a right angle.

Answer: $\vec{x}=(-4,\ 3,\ 7)+t(-5,\ 0,\ 5)$ where $t\in\mathbb{R}$

9. Given the linear system $\begin{bmatrix} k+4 & k+4 & k+4 \\ 1 & k+2 & 1 \\ 1 & 1 & k+2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

   (a) (4 marks) Find all values of k for which the system has a unique solution.

   Answer: $k\neq -4,\ -1$

   (b) (2 marks) For the value of $k=-2$ use Cramer's Rule to solve for x_1 only.

   Answer: $x_1=3/2$

10. (5 marks) If $\vec{u}$, $\vec{v}$ and $\vec{w}$ are unit vectors in $\mathbb{R}^3$ such that $\vec{u}$ and $\vec{v}$ are both orthogonal to $\vec{w}$ and that the angle between $\vec{u}$ and $\vec{v}$ is $\pi/3$ then solve for k where $\text{proj}_{\vec{v}}(\vec{u}+2\vec{v}+3\vec{w})=k\vec{v}$

Answer: $k=5/2$

11. (4 marks) Find the general equation of the plane which is orthogonal to the plane $3x-y+z=1$ and contains the line $\mathscr{L}:\ x=1+t,\ y=2+t,\ z=t$ where $t\in\mathbb{R}$.

Answer: $-2x-2y+4z=-6$

12. Given the skew lines

$$\mathscr{L}_1:(x,y,z)=(2,1,1)+t(1,1,0) \text{ where } t\in\mathbb{R}$$
$$\mathscr{L}_2:(x,y,z)=(-2,1,2)+s(-1,0,1) \text{ where } s\in\mathbb{R}.$$

   (a) (5 marks) Find the point A on line $\mathscr{L}_1$ and the point B on $\mathscr{L}_2$ which are closest to each other.

   Answer: $A(1,\ 0,\ 1)$ and $B(0,\ 1,\ 0)$

   (b) (1 mark) Find the distance between the skew lines $\mathscr{L}_1$ and $\mathscr{L}_2$.

Answer: $\sqrt{3}$

13. (5 marks) If $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^n$ such that $||\vec{u}|| = 1$, $||\vec{v}|| = 4$ and $||2\vec{u} + 3\vec{v}|| = \sqrt{172}$ find the angle θ between $\vec{u}$ and $\vec{v}$. If the angle is a special angle, give the exact value in radians.

Answer: $\theta = \pi/3$

14. Given $\vec{u}$, $\vec{v}$, $\vec{w} \in \mathbb{R}^3$ such that $(3\vec{v} \times 5\vec{w}) \cdot (-2\vec{u} - \vec{v}) = 1$.

(a) (3 marks) Find the volume of the parallelepiped generated by the vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$.

Answer: $1/30$

(b) (2 marks) Describe geometrically the span$(\{\vec{u}, \vec{v}, \vec{w}\})$. Justify.

Answer: $\mathbb{R}^3$

15. Consider the set

$$V = \{(x, y) \mid x + y = 2\}$$

under the following operations:

$$(x_1, y_1) + (x_2, y_2) = (x_1 \cdot x_2, y_1 + y_2 - 1) \qquad k \cdot (x, y) = (kx - k + 1, y^k)$$

(a) (1 mark) Find $2 \cdot (3, -1)$.

Answer: $(5, 1)$

(b) (3 marks) Show that V contains a zero vector (in the sense of a vector space).

Answer: $\vec{0} = (1, 1)$

(c) (3 marks) Does V contain an additive inverse for $\vec{v} = (-1, 3)$? If so, find it. Justify.

Answer: Does not contain an additive inverse for $\vec{v} = (-1, 3)$

16. (a) (4 marks) Show that the set of all 3×3 symmetric matrices is a subspace of $M_{3\times 3}$.

> **Answer:** Apply the subspace test (i.e. show closure under addition and scalar multiplication).

(b) (5 marks) Find a basis and the dimension of the space of all 3×3 symmetric matrices.

> **Answer:**
> $$B = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right\}$$ and
> the dimension of the space is 6.

17. (3 marks) Given the set $S = \{p_1(x),\ p_2(x),\ p_3(x)\}$ where

$$p_1(x) = 1 + 2x + x^2$$
$$p_2(x) = 2 - 3x$$
$$p_3(x) = 7 - 7x + x^2$$

find a basis for $\text{span}(S)$.

> **Answer:** $B = \{p_1(x),\ p_2(x)\}$

18. Determine whether each of the following statements is true or false. If the statement is false provide a counterexample. If the statement is true provide a proof of the statement.

(a) (3 marks) If each equation in a consistent linear system is multiplied through by a constant c, then all solutions to the new system can be obtained by multiplying solutions from the original system by c.

> **Answer:** False, provide counterexample.

(b) (3 marks) There is no 3×3 matrix for which $A^2 + I_3 = 0$.

> **Answer:** True, provide proof.

(c) (3 marks) If A and B are square matrices such that AB is the product of elementary matrices, then the homogeneous system $AX = 0$ has only the trivial solution.

> **Answer:** True, provide proof.