DAWSON COLLEGE

DEPARTMENT OF MATHEMATICS

FINAL EXAMINATION

LINEAR ALGEBRA 201-NYC-05 (Science)

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Time: 3 hours

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This examination has 10 problems.

Each problem is worth the same amount and each part of each problem is worth the same amount.

1. Consider the following system of linear equations:

$$\begin{aligned} 9x + 3y + z &= 0 \\ 3x + 2y + z &= 4 \\ 6x + 2y + z &= 1 \end{aligned}$$

a) Solve this system using the inverse of its coefficient matrix.
b) Use Gauss-Jordan elimination to solve the homogeneous system whose coefficient matrix is the augmented matrix of the given system.
c) Solve the system whose coefficient matrix is the augmented matrix of the given system and which has a particular solution $(1, -1, -1, 1)$.

2. Consider the following system:

$$\begin{aligned} kx + \quad\quad y + kz &= 1 \\ x + \quad\quad y + \ z &= 1 \\ (2-k)x + (2-k)y + \ z &= 1 \\ kx + \quad\quad y + kz &= k^2 \end{aligned}$$

For what values of k, if any, the system has: (a) no solution, (b) a unique solution, (c) infinitely many solutions.

3. Find all matrices A such that:

a) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} A = I_2$ b) $A^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 7 & 0 & 1 \end{bmatrix} = (3I_3 - 2A)^{-1}$ c) $A \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 6 & 6 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$ and A is an elementary matrix.

4. a) Solve for x: $\begin{vmatrix} sinx & cosx \\ -cosx & sinx \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ e^x & 1 & e^x \\ 1 & e^x & 0 \end{vmatrix}$

b) If $A = \begin{bmatrix} x & 1 & x \\ 1 & 1 & 1 \end{bmatrix}$ find the values of x such that $\det(AA^t) = \det(A^t A)$.

c) If A is a 3×3 matrix such that $\det(A) = 2$ find $\det(adj(2adj(A^{-1})))$.

5. Given the points $A(-3, 0, -1)$ and $B(-2, 1, 0)$.

a) Find the point C on the *YZ-plane* (i.e. the plane spanned by $\boldsymbol{j}$ and $\boldsymbol{k}$) such the points *A, B* and C are collinear .

b) Find the distance between the origin and the line passing through the points A and *B.*

c) Determine whether the points A and B are on the same side of the plane $7x + y + z = 1$.

6. a) If $\boldsymbol{u} = (0, 1, 1)$ *and* $\boldsymbol{v} = (p, 4, p)$ then find the parameter p such that the angle between vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is $\frac{\pi}{3}$.

b) Show that if $\boldsymbol{u}$ and $\boldsymbol{v}$ are vectors in $\mathbb{R}^3$ such that $\boldsymbol{u} + \boldsymbol{v}$ and $\boldsymbol{u} - \boldsymbol{v}$ have the same length then $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal.

c) If $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{w}$ are vectors in $\mathbb{R}^3$ then prove the identity:

$$(6\boldsymbol{u}) \cdot (\boldsymbol{v} \times \boldsymbol{w}) = -(\big(\boldsymbol{u} \times (3\boldsymbol{w})\big) \cdot (2\boldsymbol{v}))$$

7. Given the lines $\boldsymbol{\mathcal{L}_1}$: $(x, y, z) = (1, 3, 0) + t(4, 3, 1)$, $\boldsymbol{\mathcal{L}_2}$: $(x, y, z) = (1, 2, 3) + t(8, 6, 2)$, the plane $\boldsymbol{\mathcal{P}}$: $2x - y + 3z = 15$ and the point $A(1, 0, 7)$.

a) Show that the lines $\boldsymbol{\mathcal{L}_1}$ and $\boldsymbol{\mathcal{L}_2}$ lie in the same plane and find the general equation of this plane.

b) Find the distance between the line $\boldsymbol{\mathcal{L}_1}$ and the *Y-axis.*

c) Find the point B on the plane $\boldsymbol{\mathcal{P}}$ which is closest to the point *A.*

8. a) Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.

Determine whether the vectors A^T, A^{-1} *and* A^2 are linearly independent in $M_{2\times 2}$.

b) Let $S = \{(0, 0, x, y) \in \mathbb{R}^4 | x \leq y\}$.

Determine whether S is a vector space with standard addition and multiplication by scalar of $\mathbb{R}^4$.

c) Let $\boldsymbol{W}$ be the solution space of the homogeneous system whose coefficient matrix is:

$$\begin{bmatrix} 1 & 1 & k \\ 1 & k & 1 \\ k & 1 & 1 \end{bmatrix}$$

Find the values of k such that $\boldsymbol{W}$ is: (i) the zero space, (ii) a line through the origin, (iii) a plane through the origin.

9. Let $S = \{p(x) \in P_2 | \, p(1) = p(2)\}$.
   a) Show that S is a subspace of $P_{2.}$
   b) Find a basis and the dimension of S.
   c) Find the coordinates of the vector $p(x) = 5x^2 - 15x + 7$ relative to the basis found in part (b).

10. Determine whether the following statement is true or false. Justify your answer with a proof or a counterexample.
    a) The planes $x + y + 2z = 5$ and $(x, \; y, z\,) = (1, 2, 3\,) + (1, 1, 2)s + (3, 0, 7\,)t$ are perpendicular.
    b) If A is a matrix such that A^2 is symmetric then A is symmetric.
    c) If $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ is a basis of a vector space **V** then $\{\boldsymbol{u}, \; \boldsymbol{u} + \boldsymbol{v} + \boldsymbol{w}, \boldsymbol{u} + 2\boldsymbol{v} + 2\boldsymbol{w}\}$ is also a basis of **V**.

ANSWERS:

1. a) $(-1, 2, 3)$ b) $\{(t, -2t, -3t, t) | \; t \in \mathbb{R}\}$ c) $\{(1 + t, -1 - 2t, -1 - 3t, 1 + t) | t \in \mathbb{R}\}$

2. a) $k \neq \mp 1$ b) $k = -1$ c) $k = 1$

3. a) $\left\{ \begin{bmatrix} 1 & -1 \\ -s & 1 - t \\ s & t \end{bmatrix} \middle| \; s, t \in \mathbb{R} \right\}$ b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{7}{3} & 0 & 1 \end{bmatrix}$ c) $\begin{bmatrix} 6 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \frac{5}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & \frac{5}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

4. a) $\ln(2)$ b) 1 c) 4

5. a) $(0, 3, 2)$ b) $\sqrt{\frac{14}{3}}$ c) Yes

6. a) -1 b) Hint: Use properties of dot product. c) Hint: Use properties of scalar triple product.

7. a) $5x - 6y - 2z + 13 = 0$ b) $\frac{1}{\sqrt{17}}$ c) $\left(-\frac{1}{7}, \frac{4}{7}, \frac{37}{7}\right)$

8. a) Yes. b) No. c) $(i)\; k \neq 1 \; and \; k \neq -2$ $(ii)\; k = -2$ $(iii)\; k = 1$

9. a) Hint: Use the theorem. b) $\{\, x^2 - 3x, 1 \,\}$ c) $(5, 7)$

10. a) True b) False c) False