201-SN2-RE Final Winter 2025

1. [16 Marks] Evaluate the following limits, but DO NOT USE L'HOPITAL'S RULE.

a. $\lim_{x\to -2^-} \dfrac{x^3+1}{4-x^2} = \dfrac{(-2)^3+1}{4-(-2)^2} = \dfrac{-8+1}{4-4} = \dfrac{-7}{0}$

$\to \lim_{x\to -2^-} \dfrac{x^3+1}{(2-x)(2+x)} = \dfrac{-7}{(2+2)(2+(-2^-))} = \dfrac{-7}{4(0^-)} = \boxed{+\infty}$

b. $\lim_{\theta\to 0} \dfrac{\tan(6\theta)}{\sin(2\theta)} = \dfrac{\tan(0)}{\sin(0)} = \dfrac{0}{0}$

$= \lim_{\theta\to 0} \dfrac{\sin(6\theta)}{\cos(6\theta)} \cdot \dfrac{1}{\sin(2\theta)}$

$= \lim_{\theta\to 0} \dfrac{(6\theta)\sin(6\theta)}{6\theta} \cdot \dfrac{1}{\cos(6\theta)} \cdot \dfrac{(2\theta)\,1}{(2\theta)\sin(2\theta)}$

$= \lim_{\theta\to 0}\left(\dfrac{6\cancel{\theta}}{2\cancel{\theta}}\right) \cdot \underbrace{\lim_{\theta\to 0} \dfrac{\sin(6\theta)}{6\theta}}_{=1} \cdot \underbrace{\lim_{\theta\to 0} \dfrac{2\theta}{\sin(2\theta)}}_{=1} = \left(\dfrac{6}{2}\right) = \boxed{3}$

c. $\lim_{x\to -\infty} \dfrac{3x-1}{\sqrt{4x^2-x+1}} = \dfrac{\infty}{\infty}$

$= \lim_{x\to -\infty} \dfrac{x\left(3-\frac{1}{x}\right)}{\sqrt{x^2\left(4-\frac{1}{x}+\frac{1}{x^2}\right)}} = \lim_{x\to -\infty} \dfrac{x\left(3-\frac{1}{x}\right)}{\sqrt{x^2}\sqrt{4-\frac{1}{x}+\frac{1}{x^2}}} = \lim_{x\to -\infty} \underbrace{\dfrac{x}{|x|}}_{=-1} \dfrac{\left(3-\overset{\to 0}{\frac{1}{x}}\right)}{\sqrt{4-\underset{\to 0}{\frac{1}{x}}+\underset{\to 0}{\frac{1}{x^2}}}} = \dfrac{-1(3)}{\sqrt{4}}$

$= \boxed{\dfrac{-3}{2}}$

d. $\lim_{x\to\infty} \dfrac{\sin(2x)}{e^{2x}} = \dfrac{\sin(\infty)}{e^{\infty}}$ undefined

since $-1 \le \sin(2x) \le 1$

then $\dfrac{-1}{e^{2x}} \le \dfrac{\sin(2x)}{e^{2x}} \le \dfrac{1}{e^{2x}}$

And $\lim_{x\to\infty} \dfrac{-1}{e^{2x}} \le \underbrace{\lim_{x\to\infty} \dfrac{\sin(2x)}{e^{2x}}}_{=0} \le \lim_{x\to\infty} \dfrac{1}{e^{2x}}$

$\frac{1}{\infty} = 0$

since $\frac{-1}{\infty} = 0$

by squeeze theorem.

2. [5 Marks] Consider the function $f(x) = \frac{3x}{2x-1}$. Use the limit definition of the derivative to find $f'(x)$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\left(\frac{3(x+h)}{2(x+h)-1} - \frac{3x}{2x-1}\right)}{h} = \lim_{h\to 0} \left(\frac{3x+3h}{2x+2h-1} - \frac{3x}{2x-1}\right)\cdot\frac{1}{h}$$

$$= \lim_{h\to 0} \frac{(3x+3h)(2x-1) - 3x(2x+2h-1)}{(2x+2h-1)(2x-1)}\cdot\frac{1}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{6x^2} - \cancel{3x} + \cancel{6hx} - 3h - \cancel{6x^2} - \cancel{6hx} + \cancel{3x}}{(2x+2h-1)(2x-1)}\cdot\frac{1}{h}$$

$$= \lim_{h\to 0} \frac{-3\cancel{h}}{(2x+2h-1)(2x-1)}\cdot\frac{1}{\cancel{h}}$$

$$= \boxed{\frac{-3}{(2x-1)^2}}$$

3. [6 Marks] For which values of x is the function f discontinuous and identity the discontinuity as a hole, vertical asymptote or a jump (or gap) in the graph. (Justify your answers.)

$$f(x) = \begin{cases} \frac{x^2-4}{x^2+5x+6} & \text{if } x < -1 \\ x-1 & \text{if } -1 \le x \end{cases} = \begin{cases} \frac{(x-2)(x+2)}{(x+2)(x+3)} & \text{if } x < -1 \\ x-1 & \text{if } x \ge -1 \end{cases}$$

if $x < -1$ then $f(x) = \frac{(x-2)\boxed{(x+2)}}{(x+3)\boxed{(x+2)}}$

hole in graph at $x = -2$

Vertical asymptote at $x = -3$

$f(-2)$ is undefined but $\lim_{x\to -2^-} \frac{(x-2)(x+2)}{(x+3)(x+2)} = \frac{-4}{-2+3} = -4$ $\therefore$ $\boxed{\text{hole at } (-2,-4)}$ in graph

$f(-3)$ is undefined $\Rightarrow \frac{5}{0}$ $\boxed{\therefore \text{ V.A at } x = -2}$

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} \frac{x^2-4}{x^2+5x+6} = \frac{1-4}{1-5+6} = \frac{-3}{2}$$

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} x-1 = -1-1 = -2$$

Since $\lim_{x\to -1^-} f(x) \neq \lim_{x\to -1^+} f(x)$ then discontinuous at $x = -1$ and we have a jump (gap) in graph

4. The motion of a particle is given by $s(t) = t^3 - 5t$, where s is in meters and t in seconds. Find

a. [2 Marks] the velocity and acceleration funtions of t.

$$V(t) = s'(t) = 3t^2 - 5$$

$$a(t) = v'(t) = s''(t) = 6t$$

b. [1 Mark] the acceleration after 3 seconds and

$$a(3) = 6(3) = \boxed{18\ m/s^2}$$

c. [2 Mark] the acceleration when the velocity is 0.

$v(t) = 0$ when $3t^2 - 5 = 0$

$3t^2 = 5 \Rightarrow t^2 = 5/3 \Rightarrow \boxed{t = \pm\sqrt{5/3}}$

Since $t = -\sqrt{5/3}$ does not apply we only consider the positive time, $t = \sqrt{5/3}$

$\therefore\ a(\sqrt{5/3}) = \boxed{6\left(\frac{\sqrt{5}}{\sqrt{3}}\right)\ m/s^2}$

5. [5 Marks] Find an equation of a tangent line to the curve $y = x\sqrt{x}$ that is parallel to the line $3x - y = 4$

$y = x\sqrt{x} = x(x^{1/2}) = x^{3/2}$

$y' = \frac{3}{2}x^{1/2}$ (slope of tangent line)

$3x - y = 4$

$3x - 4 = y \Rightarrow$ slope $= 3$

If $y' = 3$ then $\frac{3}{2}\sqrt{x} = 3$

$\sqrt{x} = \frac{2}{\cancel{3}}(\cancel{3}) \Rightarrow \sqrt{x} = 2$

$\boxed{x = 4}$

at $x = 4$, $y = x\sqrt{x} = 4\sqrt{4} = 8$

$\therefore$ tangent line at $(4, 8)$ is

$$\left.\begin{aligned} y &= 3x + b \\ 8 &= 3(4) + b \\ 8 - 12 &= -4 = b \end{aligned}\right\} \quad \boxed{y = 3x - 4}$$

6. [12 Marks] Find the $\frac{dy}{dx}$: (Do not simplify your answers.)

a. $y = \left(\frac{1}{x^2} - \frac{2}{x^3}\right)(\sqrt{x} + x^{-2/3})$ $= \left(x^{-2} - 2x^{-3}\right)\left(x^{1/2} + x^{-2/3}\right)$

$y' = u'v + uv'$

$$\boxed{y' = \left(-2x^{-3} + 6x^{-4}\right)\left(x^{1/2} + x^{-2/3}\right) + \left(x^{-2} - 2x^{-3}\right)\left(\tfrac{1}{2}x^{-1/2} - \tfrac{2}{3}x^{-5/3}\right)}$$

b. $y = \frac{(x + \tan x)^3}{(1 + \cos(2x))^2}$ $= \frac{u}{v} \implies y' = \frac{u'v - uv'}{v^2}$

$$y' = \frac{3(x+\tan x)^2(1+\sec^2 x)(1+\cos(2x))^2 - (x+\tan x)^3(2)(1+\cos(2x))(0 - \sin(2x)\cdot 2)}{(1+\cos(2x))^4}$$

c. $y = x\arcsin(x^2 + x)$ $= uv$

$$\boxed{y' = 1\arcsin(x^2+x) + x \cdot \frac{1}{\sqrt{1-(x^2+x)^2}} \cdot (2x+1)}$$

7. [8 Marks] Find the y': (Do not simplify your answers.)

a. $4x^3 - 2x^2y = \sqrt{x+y}$

$$\frac{d}{dx}\left(4x^3 - 2x^2y\right) = \frac{d}{dx}\left((x+y)^{1/2}\right)$$

$$12x^2 - (4xy + 2x^2y') = \frac{1}{2}(x+y)^{-1/2}(1+y')$$

$$12x^2 - 4xy - 2x^2y' = \frac{1}{2}(x+y)^{-1/2} + \frac{1}{2}(x+y)^{-1/2}y'$$

$$12x^2 - 4xy - \frac{1}{2}(x+y)^{-1/2} = 2x^2y' + \frac{1}{2}(x+y)^{-1/2}y'$$

$$y'\left(2x^2 + \frac{1}{2}(x+y)^{-1/2}\right) = 12x^2 - 4xy - \frac{1}{2}(x+y)^{-1/2}$$

$$\boxed{y' = \frac{12x^2 - 4xy - \frac{1}{2}(x+y)^{-1/2}}{\left(2x^2 + \frac{1}{2}(x+y)^{-1/2}\right)}}$$

b. $y = x^{x\sin x}$

$$\ln y = \ln x^{x\sin x}$$

$$\ln y = x\sin x \cdot \ln x$$

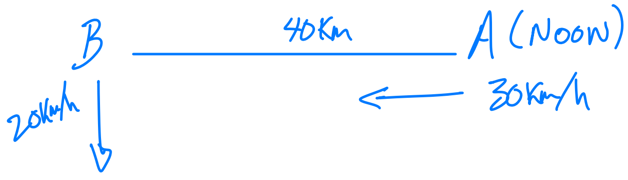
$$\frac{d}{dx}(\ln y) = \frac{d}{dx}\left(x\sin x \ln x\right)$$

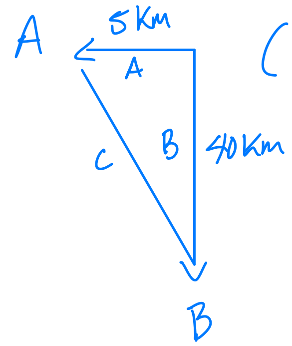
$$\frac{1}{y}\cdot y' = \sin x \cdot \ln x + x\cos x \ln x + x\sin x \cdot \frac{1}{x}$$

$$y' = y\left(\sin x \cdot \ln x + x\cos x \ln x + \cancel{x}\sin x \cdot \frac{1}{\cancel{x}}\right)$$

$$\boxed{y' = x^{x\sin x}\left(\sin x \cdot \ln x + x\cos x \ln x + \sin x\right)}$$

8. [5 Marks] At noon, ship A is 40 km east of ship B. Ship A is sailing west at 30 km/h and ship B is sailing south at 20 km/h. How fast is the distance between the ships changing at 1:30 PM?





(1:30 PM)

after $1\frac{1}{2}$ hrs A travelled 45Km

B " 40Km

$\frac{dA}{dt} = 30\text{Km/h}$

$\frac{dB}{dt} = 20\text{Km/h}$

$$A^2 + B^2 = C^2$$

$$\frac{d}{dt}(A^2+B^2) = \frac{d}{dt}(C^2)$$

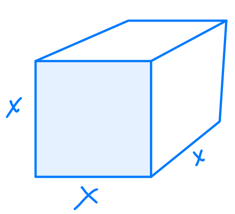
$$2A\frac{dA}{dt} + 2B\frac{dB}{dt} = 2C\frac{dC}{dt}$$

$$C = \sqrt{5^2+40^2} = \sqrt{1625}$$

$$\frac{dC}{dt} = \frac{A\frac{dA}{dt} + B\frac{dB}{dt}}{C} = \frac{(5\text{Km})\,30\text{Km/h} + (40\text{Km})\,20\text{Km/h}}{\sqrt{1625}} \approx \boxed{23.5\ \text{Km/h}}$$

9. [4 Marks] The edge of a cube was found to be 25 cm with a maximum error of 0.1 cm. Use differentials to estimate the maximum possible error in the calculationing the surface area of the cube.

$$S = 6x^2$$



$x = 25$ cm

$\Delta x = \text{error} = .1$ cm

$dx = .1$ cm

$$\frac{d}{dx}(S) = \frac{d}{dx}(6x^2)$$

$$\frac{dS}{dx} = 12x$$

$$\therefore\ dS = 12x\,dx$$

Maximum possible error for surface area of cube is ΔS

$$\Delta S \approx dS$$
$$\approx 12x\,dx$$
$$\approx 12(25\text{ cm})(0.1\text{ cm})$$

$$\boxed{\text{MAX error} \approx 30\text{ cm}^2}$$

10. [8 Marks] Differentiate the functions: (Do not simplify your answers.)

a. $f(x) = \log(x^2+4) - \dfrac{1}{\ln(x^2)+4}$ $= \dfrac{\ln(x^2+4)}{\ln 10} - (\ln(x^2)+4)^{-1}$

$$\boxed{f'(x) = \frac{1}{\ln 10}\cdot\frac{1}{x^2+4}\cdot 2x + (\ln(x^2)+4)^{-2}\cdot\frac{1}{x^2}(2x)}$$

b. $g(x) = \ln(e^{-2x}+xe^{-2x}) + 3^{\csc x}$

$$\boxed{g'(x) = \frac{1}{e^{-2x}+xe^{-2x}}\cdot\left(e^{-2x}(-2)+e^{-2x}-2xe^{-2x}\right) - 3^{\csc x}\cdot\csc x\cot x\cdot\ln 3}$$

11. [4 Marks] Use l'Hopital's rule to find the limit: $\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)$. $= \frac{1}{0^+} - \frac{1}{1^+-1} = \infty - \infty$

$$\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{e^x-1}\right) = \lim_{x\to 0^+}\left(\frac{1(e^x-1)-x}{x(e^x-1)}\right) = \lim_{x\to 0^+}\frac{e^x-1-x}{xe^x-x} = \frac{0}{0}$$

$$\overset{(H)}{=} \lim_{x\to 0^+}\frac{e^x-1}{e^x+xe^x-1} = \frac{1-1}{1^++0^+-1} = \frac{0}{0^+}$$

$$\overset{(H)}{=} \lim_{x\to 0^+}\frac{e^x}{e^x+e^x+xe^x} = \frac{1}{1+1+0} = \boxed{1/2}$$

12. Consider the function $f(x) = x\sqrt{4-x^2}$, where $f'(x) = -2\dfrac{x^2-2}{\sqrt{4-x^2}}$ and $f''(x) = 2\dfrac{x(x^2-6)}{\sqrt{(4-x^2)^3}}$.

For the function f, find (if any)

a. [1 Marks] The x and y intercepts.

$f(0) = 0 \implies y \text{ intercept} = 0$

$0 = f(x) \implies x\sqrt{4-x^2} = 0$

$\implies \boxed{x = 0 \text{ or } x = \pm 2}$ x intercepts

b. [2 Marks] The vertical and horizontal asymptotes.

No vertical asymptotes

No horizontal asymptotes because Domain of f is $[-2, 2]$.

c. [3 Marks] The relative extrema and the interval(s) where f is increasing and decreasing.

Critical numbers:

$f'(x) = 0 \implies \dfrac{-2(x^2-2)}{\sqrt{4-x^2}} = 0 \implies x^2-2=0 \implies \boxed{x = \pm\sqrt{2}}$

interval	$(-2, -\sqrt{2})$	$-\sqrt{2}$	$(-\sqrt{2}, \sqrt{2})$	$\sqrt{2}$	$(\sqrt{2}, 2)$
Sign of $f'(x)$	$-$	relative minimum	$+$	relative maximum	$-$
$f(x)$	$\searrow$		$\nearrow$		$\searrow$

$f(x)$ is decreasing on $(-2, -\sqrt{2})$ and $(\sqrt{2}, 2)$.

" " increasing on $(-\sqrt{2}, \sqrt{2})$.

Relative minimum at $(-\sqrt{2}, -2)$ and

" maximum at $(\sqrt{2}, 2)$.

$f(-\sqrt{2}) = -\sqrt{2}\sqrt{4-(-\sqrt{2})^2} = -\sqrt{2}\sqrt{4-2} = -\sqrt{2}\sqrt{2} = -2$

$f(\sqrt{2}) = \sqrt{2}\sqrt{4-(\sqrt{2})^2} = \sqrt{2}\sqrt{4-2} = \sqrt{2}\sqrt{2} = 2$

d. [3 Marks] The points of inflection and intervals of concavity.

$f''(x) = 0 \Rightarrow \dfrac{2x(x^2-6)}{\sqrt{(4-x^2)^3}} = 0 \Rightarrow \underset{0}{2x}\underset{0}{(x^2-6)} = 0 \Rightarrow \boxed{x=0}$ or $\boxed{\underbrace{x = \pm\sqrt{6}}_{\text{Not in domain}}}$

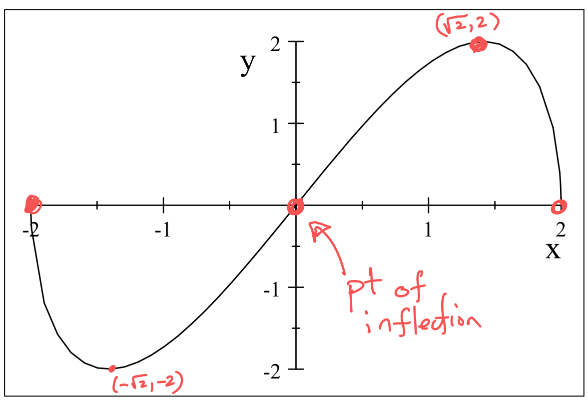
Intervals	$(-2, 0)$	0	$(0, 2)$
Sign of $f''(x)$	$+$	Point of inflection	$-$
$f(x)$	$\cup$		$\cap$

There is a point of inflection at $(0,0)$

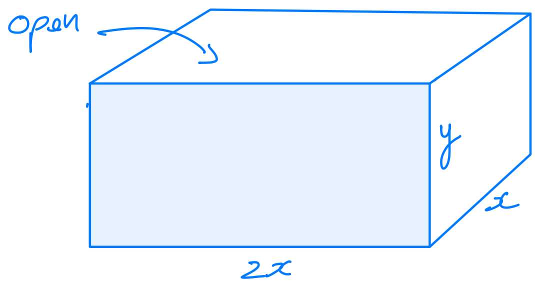
$f(x)$ is concave upward on $(-2,0)$ and

" " downward on $(0,2)$.

e. [3 Marks] Sketch the graph of *f*, labelling all the important points that you have found.



13. [5 Marks] A rectangular storage container with an open top is to have a volume of 4 m^3. The length of its base is twice the width. Material for the base costs \$10 per square meter. The cost of the sides is \$6 per square meter. Find the cost of the material for the cheapest such container.



$y = 2(\sqrt[3]{.8})^{-2} \approx 2.32079$

$\boxed{\text{Cost} = 20(\sqrt[3]{.8})^2 + 32(\sqrt[3]{.8})^{-1} \approx \$51.71}$

$V = 2x^2y = 4m^3$ (Volume) $\Longrightarrow y = \frac{4}{2x^2}$

$y = 2x^{-2}$

$C = \$10(2x^2) + \$6(2xy + 2(2xy))$

$C = 20x^2 + 16xy$

$C = 20x^2 + 16x(2x^{-2})$

$C = 20x^2 + 32x^{-1}$

$C' = 40x - 32x^{-2} = 0$

$\Longrightarrow 40x = \frac{32}{x^2} \Longrightarrow x^3 = \frac{32}{40} = \frac{4}{5}$

$C'' = 40 + 64x^{-3}$

$C'' > 0$ so rel. min

$x = \sqrt[3]{\frac{4}{5}}$ Critical number

$x \approx 0.928318$

14. [5 Marks] Find $f(x)$ given that $f''(x) = 2e^x + 3\sin x$, $f(0) = 0$ and $f(\pi) = 0$.

$f'(x) = \int (2e^x + 3\sin x)\,dx$

$f'(x) = 2e^x - 3\cos x + C_1$

$f(x) = \int (2e^x - 3\cos x + C_1)\,dx = 2e^x - 3\sin x + C_1x + C_2$

$f(0) = 2e^0 - 3\sin 0 + C_1(0) + C_2 = 0$

$\Longrightarrow 2 - 0 + 0 + C_2 = 0 \Longrightarrow \boxed{C_2 = -2}$

$f(\pi) = 2e^\pi - 3\sin\pi + C_1\pi - 2 = 0$

$\Longrightarrow 2e^\pi - 0 + C_1\pi - 2 = 0$

$\Longrightarrow C_1\pi = 2 - 2e^\pi \Longrightarrow \boxed{C_1 = \frac{2-2e^\pi}{\pi}}$

$$\boxed{f(x) = 2e^x - 3\sin x + \left(\frac{2-2e^\pi}{\pi}\right)x - 2}$$

15. [5 Marks] Evaluate $\int \frac{x^3}{\sqrt{x^2+4}}\,dx$.

let $u = x^2+4 \implies du = 2x\,dx \implies \frac{du}{2} = x\,dx$

$x^2 = u-4$

$$\int \frac{x^3}{\sqrt{x^2+4}}\,dx = \int \frac{x^2\,x}{\sqrt{x^2+4}}\,dx = \int \frac{u-4}{\sqrt{u}}\,\frac{du}{2} = \frac{1}{2}\int\left(\frac{u}{\sqrt{u}} - \frac{4}{\sqrt{u}}\right)du$$

$$= \frac{1}{2}\int\left(u^{1/2} - 4u^{-1/2}\right)du$$

$$= \frac{1}{2}\left(\frac{u^{3/2}}{3/2} - \frac{4u^{1/2}}{1/2}\right) + C$$

$$= \frac{1}{2}\left(\frac{2}{3}(x^2+4)^{3/2} - 8\sqrt{x^2+4}\right) + C$$

$$= \boxed{\frac{1}{3}(x^2+4)^{3/2} - 4\sqrt{x^2+4} + C}$$