Dawson College
Department of Mathematics
Differential Calculus
201-SN2-RE
Final Examination
Winter 2026

Student Name:_____

Student I.D#:_____

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Date: Thursday, May 21, 2026,

Time: 2:00 pm to 5:00 pm

Instructions:

- Print your name and ID in the provided space.
- This exam contains 15 pages (including this cover page) and 15 questions for a total of 100 points.
- Check to see if any pages are missing.
- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page, and please indicate that you have done so.
- You are only permitted to use the **Sharp EL-531**** calculator.
- This examination booklet must be returned intact.
- Good luck!

Question	Points	Score
1	12	
2	5	
3	5	
4	8	
5	5	
6	4	
7	12	
8	6	
9	5	
10	5	
11	5	
12	11	
13	5	
14	4	
15	8	
Total	**100**	

1. (12 points) Evaluate the following limits **without** using L'Hopital's rule.

a) $\lim_{x\to 3} \dfrac{\sqrt{x^2-3x+4}-2}{3-x}$ $= \dfrac{\sqrt{9-9+4}-2}{3-3} = \dfrac{0}{0}$

$$= \lim_{x\to 3} \left(\frac{\sqrt{x^2-3x+4}-2}{3-x}\right)\left(\frac{\sqrt{x^2-3x+4}+2}{\sqrt{x^2-3x+4}+2}\right)$$

$$= \lim_{x\to 3} \frac{x^2-3x+4-4}{(3-x)(\sqrt{x^2-3x+4}+2)}$$

$$= \lim_{x\to 3} \frac{-x(3-x)}{(3-x)(\sqrt{x^2-3x+4}+2)} = \frac{-3}{\sqrt{4}+2} = \boxed{\frac{-3}{4}}$$

b) $\lim_{x\to 0} \dfrac{\sin^2(4x)}{x\sin(3x)} = \dfrac{0}{0}$

$$= \lim_{x\to 0} \frac{4\sin(4x)}{4x}\,\frac{\sin(4x)}{\sin(3x)}\,\frac{(3x)}{(3x)}\,\frac{(4x)}{4x}$$

$$= \lim_{x\to 0} \left(\frac{\sin(4x)}{(4x)}\right)\left(\frac{\sin(4x)}{(4x)}\right)\left(\frac{(3x)}{\sin(3x)}\right)\frac{4(4x)}{(3x)} = \boxed{\frac{16}{3}}$$

(each of the first three factors $\to 1$)

c) $\lim_{x\to 0} x^2\cos^2\left(\dfrac{\pi}{x^2}\right)$ $= 0\cdot\cos^2\left(\frac{\pi}{0}\right)$ (undefined)

$0 \le \cos^2(\theta) \le 1$ for all θ.

$0 \le \cos^2\left(\frac{\pi}{x^2}\right) \le 1$

$x^2(0) \le x^2\cos^2\left(\frac{\pi}{x^2}\right) \le x^2$

$$\lim_{x\to 0} 0 \le \lim_{x\to 0} x^2\cos^2\left(\frac{\pi}{x^2}\right) \le \lim_{x\to 0} x^2$$

$0 \qquad\qquad\qquad\qquad 0$

$\boxed{\lim_{x\to 0} x^2\cos^2\left(\frac{\pi}{x^2}\right) = 0}$ by squeeze theorem

d) $\lim_{x\to 4^+} \dfrac{2x-3}{16-x^2}$ $= \dfrac{2(4)-3}{16-16} = \dfrac{5}{0}$

$= \lim_{x\to 4^+} \dfrac{2x-3}{(4-x)(4+x)} = \dfrac{5}{(4-4^+)(4+4)} = \dfrac{5}{0^-(8)} = \boxed{-\infty}$

2. (5 points) Let $f(x) = \dfrac{x}{x+3}$. Use the **limit definition of the derivative** to find $f'(x)$.

(No marks will be given if you do not use the limit definition of the derivative.)

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \left(\frac{x+h}{x+h+3} - \frac{x}{x+3} \right) \Big/ h = \frac{0}{0}$$

$$= \lim_{h\to 0} \left(\frac{x+h}{x+h+3}\left(\frac{x+3}{x+3}\right) - \frac{x}{x+3}\left(\frac{x+h+3}{x+h+3}\right) \right) \Big/ h$$

$$= \lim_{h\to 0} \frac{x^2+3x+hx+3h-(x^2+xh+3x)}{(x+h+3)(x+3)\,h}$$

$$= \lim_{h\to 0} \frac{3h}{(x+h+3)(x+3)\,h}$$

$$= \boxed{\frac{3}{(x+3)^2}}$$

3. (5 points) Find the values of a and b so that the function is continuous everywhere.

$$f(x) = \begin{cases} 2x^2 - 11 & \text{if} \quad x \leq -3 \\ ax + b & \text{if} \quad -3 < x < 2 \\ -7 - ax & \text{if} \quad x \geq 2 \end{cases}$$

$f(-3) = 2(-3)^2 - 11 = 7$

$\lim_{x \to -3^-} 2x^2 - 11 = 7$

$\lim_{x \to -3^+} ax + b = -3a + b$

$-3a + b = 7$

If f is continuous at $x = -3$ then $-3a + b = 7$

$f(2) = -7 - a(2) = -2a - 7$

$\lim_{x \to 2^-} ax + b = 2a + b$

$\lim_{x \to 2^+} -7 - ax = -2a - 7$

$2a + b = -2a - 7$

$4a + b = -7$

If f is continuous at $x = 2$ then $4a + b = -7$

Solve system

$-3a + b = 7$

$4a + b = -7$ subtract Eq① from Eq②

$$\begin{array}{r} 4a + b = -7 \\ +3a - b = -7 \\ \hline 7a + 0 = -14 \end{array} \Rightarrow \boxed{a = -2} \Rightarrow -3(-2) + b = 7$$

$b = 7 - 6$

$\boxed{b = 1}$

∴ if $a = -2$ and $b = 1$ then the function is continuous everywhere.

4. (8 points) Consider the following function.

$$f(x) = \begin{cases} \dfrac{x^2-9}{(x+1)(x+3)} & \text{if } x < 1 \\ \dfrac{x+3}{x^2+1} & \text{if } x \geq 1 \end{cases}$$

a) [3 Marks] Where is the function discontinuous? Justify your answers.

If $x+1=0$ and $x+3=0$ then f is discontinuous

$x=-1$ and $x=-3$ because division by zero

$\lim_{x\to 1^-} \frac{x^2-9}{(x+1)(x+3)} = \frac{1-9}{2(4)} = -1$

$\lim_{x\to 1^+} \frac{x+3}{x^2+1} = \frac{1+3}{1+1} = \frac{4}{2} = 2$

$\lim_{x\to 1} f(x)$ DNE

$\therefore f$ is discontinuous at $x=1$

b) [3 Marks] Describe each discontinuity as a hole (removable discontinuity), a vertical asymptote (infinite discontinuity) or a jump (gap). Justify your answers.

$f(x) = \frac{(x+3)(x-3)}{(x+3)(x+1)}$

Hole in graph at $x=-3$

Vertical asymptote at $x=-1$ $\left(\frac{\#}{0}\right)$

Jump at $x=1$

Since $\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$

c) [2 Marks] Find any horizontal asymptote(s) of the function, if any.

$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^2-9}{(x+3)(x+1)} = \lim_{x\to-\infty} \frac{x^2-9}{x^2+4x+3} = \lim_{x\to-\infty} \frac{x^2\left(1-\frac{9}{x^2}\right)}{x^2\left(1+\frac{4}{x}+\frac{3}{x^2}\right)} = 1$ (HA)

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x+3}{x^2+1} = \frac{\infty}{\infty} \Rightarrow \lim_{x\to\infty} \frac{x\left(1+\frac{3}{x}\right)}{x^2\left(1+\frac{1}{x^2}\right)} = \frac{1+0}{\infty(1+0)} = \frac{1}{\infty} = 0$ (HA)

5. (5 points) Given the function $f(x) = (x^3+3)^3\left(\frac{x^2+1}{2}\right)^4$.

a) [3 Marks] Find $f'(x)$. **(Do not simplify the answer.)**

$$f'(x) = u'v + uv' = 3(x^3+3)^2(3x^2)\left(\frac{x^2+1}{2}\right)^4 + (x^3+3)^3(4)\left(\frac{x^2+1}{2}\right)^3\left(\frac{2x}{2}\right)$$

$$f'(-1) = 3(-1+3)^2(3)\left(\frac{1+1}{2}\right)^4 + (-1+3)^3(4)\left(\frac{1+1}{2}\right)^3(-1)$$

$$= 3(4)(3)(1) + 2^3(4)(1)(-1)$$

$$= 36 - 32$$

$$= \boxed{4 \quad (\text{slope})}$$

$$(x, y) = (-1, 8)$$

$$y = f(-1) = (-1+3)^3\left(\frac{(-1)^2+1}{2}\right)^4 = 2^3(1) = 8$$

b) [2 Marks] Find an equation of the tangent line of the function at $x = -1$.

Tangent line

$$y = 4x + b$$

$$8 = 4(-1) + b$$

$$b = 12$$

$$\boxed{y = 4x + 12}$$

6. (4 points) Differentiate the function $y = \dfrac{\cos\left(\sqrt{4x+1}\right)}{x^2+2}$. (**The answer does not need to be simplified.**)

$$y' \quad \frac{v - uv'}{v^2} = \frac{\sin\left(\sqrt{4x+1}\right) \cdot \frac{4}{2\sqrt{4x+1}}\left(x^2+2\right) - \cos\left(\sqrt{4x+1}\right) \cdot 2x}{\left(x^2+2\right)^2}$$

7. (12 points) Differentiate the functions. (**Answers do not need to be simplified.**)

a) $y = xe^{x^3} - \log(x^2+4x) + 3^{\csc x}$

$\log(x^2+4x) = \frac{\ln(x^2+4x)}{\ln 10}$

$$y' = e^{x^3} + x e^{x^3} \cdot 3x^2 - \frac{1}{\ln 10} \cdot \frac{1}{x^2+4x}(2x+4) + 3^{\csc x} \cdot -\csc x \cot x \cdot \ln 3$$

b) $f(x) = (\arcsin(\ln(x)))^2$

$$f'(x) = 2(\arcsin(\ln x)) \cdot \frac{1}{\sqrt{1-(\ln x)^2}} \cdot \frac{1}{x}$$

c) $y = x^{\tan x}$ $\Rightarrow \ln y = \ln\left(x^{\tan x}\right) \Rightarrow \ln y = \tan x \cdot \ln x$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(\tan x \cdot \ln x)$$

$$\frac{1}{y} \cdot y' = \sec^2 x \cdot \ln x + \frac{\tan x}{x} \Rightarrow y' = y \cdot \left(\sec^2 x \cdot \ln x + \frac{\tan x}{x}\right)$$

$$\boxed{y' = x^{\tan x}\left(\sec^2 x \cdot \ln x + \frac{\tan x}{x}\right)}$$

8. (6 points) Use l'Hopital's rule to find the limits:

a) $\lim_{x\to 0} \dfrac{xe^x}{(x+1)\sin(\pi x)} = \dfrac{0}{0}$

$\overset{(H)}{=} \lim_{x\to 0} \dfrac{1e^x + xe^x}{\sin(\pi x) + (x+1)\cos(\pi x)\cdot\pi}$

$= \dfrac{e^0 + 0}{\sin(0) + (0+1)\cos(0)\cdot\pi}$

$= \boxed{\dfrac{1}{\pi}}$

b) $\lim_{x\to 3} \left(\dfrac{2x^2 - 5x - 3}{\ln(2x-5)}\right) = \dfrac{18-15-3}{\ln(1)} = \dfrac{0}{0}$

$\overset{(H)}{=} \lim_{x\to 3} \dfrac{4x-5}{\frac{1}{2x-5}(2)} = \dfrac{4(3)-5}{2/1} = \boxed{\dfrac{7}{2}}$

9. (5 points) Find $\dfrac{dy}{dx}$ by implicit differentiation of $(x+y)^3 = x^3 + y^3$, and evaluate the derivative at the point $(-1, 1)$.

$\dfrac{d}{dx}(x+y)^3 = \dfrac{d}{dx}(x^3+y^3)$

$3(x+y)^2(1+y') = 3x^2 + 3y^2\cdot y'$

$3(x+y)^2 + 3(x+y)^2\cdot y' = 3x^2 + 3y^2\cdot y'$

$3(x+y)^2\cdot y' - 3y^2\cdot y' = 3x^2 - 3(x+y)^2$

$y' = \dfrac{3x^2 - 3(x+y)^2}{3(x+y)^2 - 3y^2}$

$= \dfrac{3x^2 - (3x^2 + 6xy + 3y^2)}{3x^2 + 6xy + 3y^2 - 3y^2}$

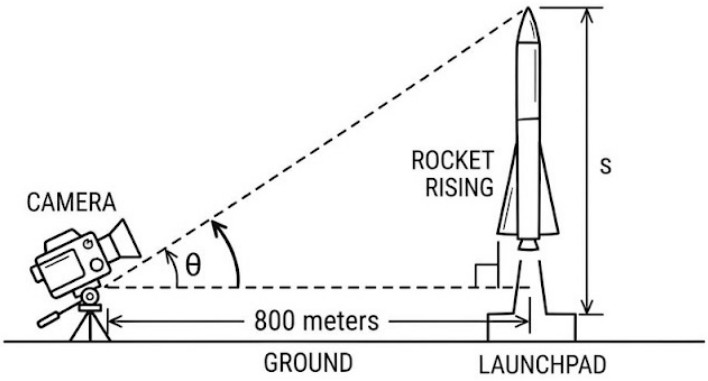
$= \dfrac{-6xy - 3y^2}{3x^2 + 6xy} = \boxed{\dfrac{-2xy - y^2}{x^2 + 2xy}}$

at $(-1, 1)$

$y' = \dfrac{-2(-1)(1) - 1^2}{(-1)^2 + 2(-1)(1)}$

$= \dfrac{2-1}{1-2} = \boxed{-1}$

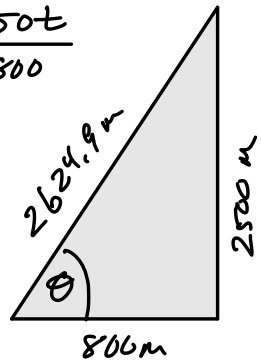
10. (5 points) A rocket is rising vertically in such a way that its height above the ground t seconds after lift-off is $s = 25t^2$ (measured in meters). A camera at ground level, 800 m from the launch pad, is filming the rocket as it ascends. If the camera stays focused on the rocket throughout its ascent, how fast is the angle of elevation of the camera changing 10 seconds after lift-off?



$$\tan\theta = \frac{s}{800} = \frac{25t^2}{800}$$

$$\frac{d}{dt}(\tan\theta) = \frac{d}{dt}\left(\frac{25t^2}{800}\right)$$

$$\sec^2\theta \frac{d\theta}{dt} = \frac{50t}{800}$$



After 10 seconds

$s(10) = 25(10^2) = 2500$ m

$a^2 + b^2 = c^2$

$800^2 + 2500^2 = c^2$

$c = \sqrt{800^2 + 2500^2} \approx 2624.9$ m

$\sec^2\theta = \left(\frac{\text{hyp}}{\text{adj}}\right)^2 = \left(\frac{2624.9}{800}\right)^2 = 10.77$

$$\frac{d\theta}{dt} = \frac{50(10)}{800} \Big/ \sec^2\theta$$

$$= \frac{5}{8} / 10.77$$

$$\frac{d\theta}{dt} \approx 0.058 \ \frac{\text{radians}}{\text{sec.}}$$

11. (5 points) Find the absolute extrema of the function $f(x) = \dfrac{2x}{x^2+1}$ on the closed interval $[-2, 2]$.

$$f'(x) = \frac{u'v - uv'}{v^2} = \frac{2(x^2+1) - 2x(2x)}{(x^2+1)^2} = \frac{2x^2+2-4x^2}{(x^2+1)^2} = \frac{-2x^2+2}{(x^2+1)^2}$$

$$f'(x) = 0 \Rightarrow -2x^2+2 = 0 \Rightarrow 2x^2 = 2 \Rightarrow \boxed{x = \pm 1} \quad \text{C.N.}$$

Check

$$f(-2) = \frac{2(-2)}{(-2)^2+1} = \frac{-4}{5}$$

$$f(-1) = \frac{2(-1)}{(-1)^2+1} = \boxed{-1} \text{ (abs minimum)}$$

$$f(1) = \frac{2(1)}{1+1} = \boxed{1} \text{ (abs maximum)}$$

$$f(2) = \frac{2(2)}{2^2+1} = \frac{4}{5}$$

12. (11 points) Analyze and sketch the graph of $f(x) = \dfrac{x^2-4}{(x-1)^2}$ where $f'(x) = \dfrac{8-2x}{(x-1)^3}$ and $f''(x) = \dfrac{4x-22}{(x-1)^4}$. Find (if any)

a) [1 marks] the domain of f.

$x-1 \neq 0 \Rightarrow \boxed{x \neq 1}$

b) [1 marks] the x and y intercepts.

let $x=0 \Rightarrow f(0) = \dfrac{0-4}{(0-1)^2} = \boxed{-4}$ (y intercept)

let $y=0 \Rightarrow 0 = x^2-4 \Rightarrow \boxed{x = \pm 2}$ x intercepts

c) [2 marks] the vertical and horizontal asymptotes.

Vertical asymptotes

$f(x) = \dfrac{x^2-4}{(x-1)^2} \Rightarrow$ V.A at $\boxed{x=1}$

Horizontal asy

$\lim\limits_{x\to\infty} \dfrac{x^2-4}{(x-1)^2} = \dfrac{\infty}{\infty} \overset{(H)}{\Rightarrow} \lim\limits_{x\to\infty} \dfrac{2x}{2(x-1)} = \boxed{1}$

$\lim\limits_{x\to-\infty} \dfrac{x^2-4}{(x-1)^2} = \dfrac{\infty}{\infty} \overset{(H)}{\Rightarrow} \lim\limits_{x\to-\infty} \dfrac{2x}{2(x-1)} = \boxed{1}$

$$f(x) = \frac{x^2-4}{(x-1)^2} \qquad f'(x) = \frac{8-2x}{(x-1)^3} \qquad f''(x) = \frac{4x-22}{(x-1)^4}$$

d) [3 marks] any relative extrema and interval(s) where f is increasing or decreasing. Create a table to display the analysis.

Critical numbers (C.N)

$f'(x) = 0 \Rightarrow 8-2x = 0 \Rightarrow \boxed{x = 4}$

Domain	$(-\infty, 1)$	VA 1	$(1, 4)$	CN 4	$(4, \infty)$
test value	0		2		100
Sign $f'(x)$	$-$		$+$	local MAX	$-$
$f(x)$	↘		↗		↘

increasing on $(1, 4)$

decreasing on $(-\infty, 1)$ and $(4, \infty)$

local MAximum at $x = 4$ and $y = \frac{4^2-4}{(4-1)^2} = \frac{12}{3^2}$

$(4, 4/3)$

e) [3 marks] any point(s) of inflection and intervals of concavity. Create a table to display the analysis.

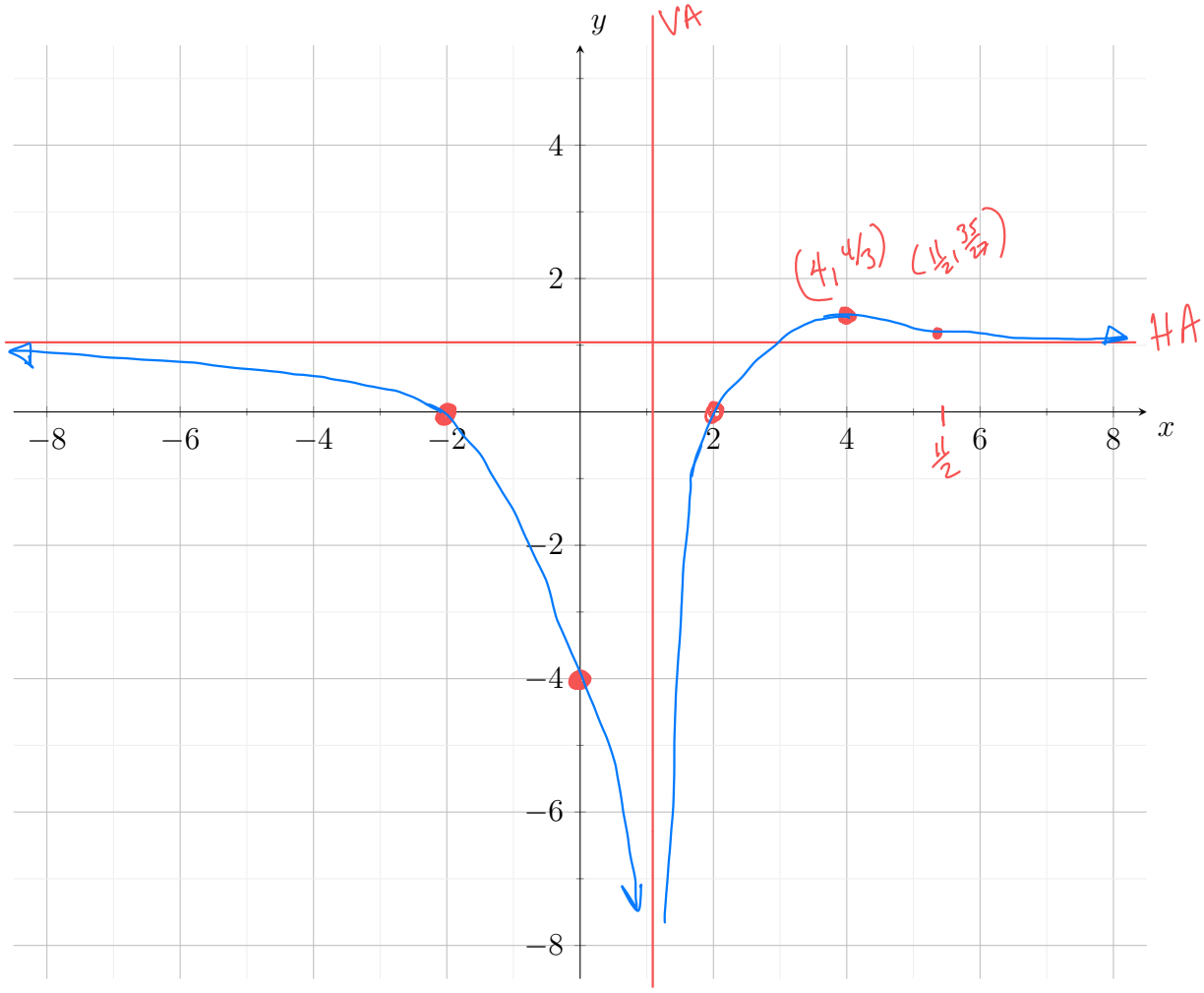
points of inflection

$f''(x) = 0 \Rightarrow 4x-22 = 0 \Rightarrow x = \frac{22}{4} = \frac{11}{2}$

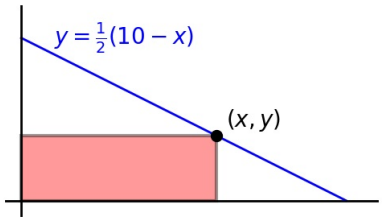
$y = \frac{(\frac{11}{2})^2-4}{(\frac{11}{2}-1)^2} = \boxed{\frac{35}{27} \approx 1.3}$

Domain	$(-\infty, 1)$	VA 1	$(1, 5.5)$	$\frac{11}{2}$	$(5.5, \infty)$
test value	0		2		100
Sign $f''(x)$	$-$		$-$		$+$
$f(x)$	∩		∩		∪

f) [1 marks] Sketch the graph of f, labelling all significant points, holes in graph, vertical or horizontal asymptotes found.



13. (5 points) Find the maximum area of a rectangle bounded by the x- and y-axes and the graph of $y = (10 - x)/2$. What length and width should the rectangle have so that the area is a maximum?



Area of rectangle $= xy$

$A = x\left(\frac{10-x}{2}\right)$

$A = \frac{10x - x^2}{2}$

$A' = \frac{10 - 2x}{2}$

$A' = 0 \Rightarrow 10 - 2x = 0$

$x = 5$ (cm)

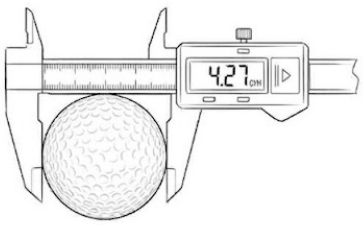
$A'' = -1 < 0$ $\Rightarrow$ local maximum

if $x = 5$ then $y = \frac{10-5}{2} = \frac{5}{2}$

$(x, y) = (5, 5/2)$

gives maximum Area of rectangle.

14. (4 points) The diameter of a golf ball is given as 4.27 cm with a possible **error** of ± 0.01 cm.



a) [2 marks] The volume of a sphere is $V = \frac{\pi d^3}{6}$ where d is the diameter of the sphere. What is the differential of the volume in terms of the differential of the diameter?

$d' = dd$

$\frac{dV}{dd} = \frac{\pi}{6}(3d^2) \Rightarrow dV = \frac{\pi}{2} d^2 d'$

b) [2 marks] Use differentials to estimate the maximum error in the calculated volume of this golf ball? What is the relative error in calculating the volume?

$d' = \pm 0.01$ cm

$dV = \frac{\pi}{2}(4.27\text{ cm})^2(\pm 0.01\text{ cm})$

$\approx 0.286\text{ cm}^3$

relative error

$\frac{\Delta V}{V} \approx \frac{dV}{V} = \frac{0.286\text{ cm}^3}{\frac{\pi(4.27)^3}{6}}$

≈ 0.007

15. (8 points) Find the indefinite integrals:

a) $\int \left(\frac{x^2 - x + 1}{\sqrt{x}} - \sec^2 x + e \right) dx$

$$= \int \left(\frac{x^2}{\sqrt{x}} - \frac{x}{\sqrt{x}} + \frac{1}{\sqrt{x}} - \sec^2 x + e \right) dx$$

$$= \int \left(x^{3/2} - x^{1/2} + x^{-1/2} - \sec^2 x + e \right) dx$$

$$= \boxed{\frac{2x^{5/2}}{5} - \frac{2x^{3/2}}{3} + 2\sqrt{x} - \tan x + ex + C}$$

b) $\int \left(\tan x + \frac{\cos(\sqrt{x})}{\sqrt{x}} \right) dx$

$$= \int \tan x \, dx + \int \frac{\cos(\sqrt{x})}{\sqrt{x}} \, dx$$

$$= \int \frac{\sin x}{\cos x} \, dx + \int \frac{\cos(\sqrt{x})}{\sqrt{x}} \, dx$$

$u = \cos x$
$du = -\sin x \, dx$
$-du = \sin x \, dx$

$w = \sqrt{x} = x^{1/2}$
$dw = \frac{1}{2} x^{-1/2} dx$
$2dw = \frac{1}{\sqrt{x}} dx$

$$= \int \frac{-du}{u} + \int \cos(w) \, 2dw$$

$$= -\ln|u| + 2\sin(w) + C$$

$$= \boxed{-\ln|\cos x| + 2\sin(\sqrt{x}) + C}$$