SN3-Final Exam-Winter 2026-A (Solutions)

(1) (**5 marks**) Use only the limit definition of the definite integral (Riemann Sum) to evaluate:

$$\int_1^4 (-x^2+3x+2)dx$$

Solution:

Partition $[1,4]$ into n subintervals:

$$\Delta x = \frac{4-1}{n} = \frac{3}{n}, \quad x_i = 1 + \frac{3i}{n}$$

Riemann sum:

$$\int_1^4 (-x^2+3x+2)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} \left[-\left(1+\frac{3i}{n}\right)^2 + 3\left(1+\frac{3i}{n}\right) + 2 \right] \frac{3}{n}$$

$$f(x_i) = -1 - \frac{6i}{n} - \frac{9i^2}{n^2} + 3 + \frac{9i}{n} + 2 = 4 + \frac{3i}{n} - \frac{9i^2}{n^2}$$

Thus:

$$\int_1^4 (-x^2+3x+2)dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left(4 + \frac{3i}{n} - \frac{9i^2}{n^2}\right) \frac{3}{n}$$

$$= \lim_{n\to\infty} \left[12 + \frac{9}{n^2} \sum i - \frac{27}{n^3} \sum i^2 \right]$$

Use formulas:

$$= \lim_{n\to\infty} \left[12 + \frac{9}{n^2} \cdot \frac{n(n+1)}{2} - \frac{27}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} \right]$$

$$= 3\left[4 + \frac{3}{2} - 3\right] = 3 \cdot \frac{5}{2} = \boxed{\frac{15}{2}}$$

(2) (**5 marks**) If $\int_2^6 4f(x)\,dx = 10$ and $\int_5^6 f(x)\,dx = 2$, find $\int_5^2 (3f(x)-2)\,dx$.

Solution:

$$\int_2^6 4f(x)\,dx = 10 \Rightarrow \int_2^6 f(x)\,dx = \frac{10}{4} = \frac{5}{2}$$

$$\int_2^5 f(x)\,dx = \frac{5}{2} - 2 = \frac{1}{2}$$

$$\int_5^2 f(x)\,dx = -\frac{1}{2}$$

$$\int_5^2 (3f(x)-2)\,dx = 3\int_5^2 f(x)\,dx - 2x|_5^2 = 3(-\frac{1}{2} - 2(2-5) =$$

$$= -\frac{3}{2} + 6 = \boxed{\frac{9}{2}}$$

(3) (**5 marks**) Use FTC to find $f(x)$ where

$$\int_e^x \frac{f(t)}{2t^2+1}\,dt = \ln(3x) - \frac{x}{2e}$$

Solution:
Differentiate both sides:

$$\frac{f(x)}{2x^2+1} = \frac{1}{x} - \frac{1}{2e}$$

$$\boxed{f(x) = (2x^2+1)\left(\frac{1}{x} - \frac{1}{2e}\right)}$$

(4) (**5 marks**) Find the average value of $g(x) = \dfrac{3x}{\sqrt{1+2x}}$ over $[0,4]$.

Solution:

$$g_{avg} = \frac{1}{4}\int_0^4 \frac{3x}{\sqrt{1+2x}}dx$$

Let $u = 1+2x$:

$$du = 2dx, \quad x = \frac{u-1}{2}$$

$$g_{avg} = \frac{1}{4}\int_1^9 \frac{3(u-1)}{2\sqrt{u}}\cdot\frac{du}{2} = \frac{3}{16}\int_1^9 \frac{u-1}{\sqrt{u}}du$$

$$= \frac{3}{16}\int_1^9 (u^{1/2} - u^{-1/2})du = \frac{3}{16}\left[\frac{2}{3}u^{3/2} - 2u^{1/2}\right]_1^9$$

$$= \frac{3}{16}\left[(18-6) - (\frac{2}{3}-2)\right] = \frac{3}{16}\cdot\frac{40}{3} = \boxed{\frac{5}{2}}$$

(5) (**18 marks**)

(a) $\displaystyle\int 6x\arctan(3x)dx$

Solution:
Integration by parts:

$$u = \arctan(3x), \quad dv = 6xdx$$

$$du = \frac{3}{1+9x^2}dx, \quad v = 3x^2$$

$$I = 3x^2\arctan(3x) - \int \frac{9x^2}{1+9x^2}dx = 3x^2\arctan(3x) - \int \frac{1+9x^2-1}{1+9x^2}dx$$

$$= \boxed{3x^2\arctan(3x) - x + \frac{1}{3}\arctan(3x) + C}$$

(b) $\int \tan^5(3t)\sec^3(3t)dt$

Solution:

$$I = \int \tan^4(3t)\sec^2(3t)\ \tan(3t)\sec(3t)dt = \int (\sec^2(3t)-1)^2\sec^2(3t)\ \tan(3t)\sec(3t)dt$$

Let $u = \sec(3t) \ \rightarrow \ du = 3\sec(3t)\tan(3t)\,dt$:

$$I = \frac{1}{3}\int (u^2-1)^2u^2du = \frac{1}{3}\int (u^6-2u^4+u^2)\,du = \frac{1}{21}u^7 - \frac{2}{15}u^5 + \frac{1}{9}u^3 + C$$

$$= \boxed{\frac{1}{21}\sec^7(3t) - \frac{2}{15}\sec^5(3t) + \frac{1}{9}\sec^3(3t) + C}$$

(c) $\int \dfrac{3dx}{e^x\sqrt{1+e^{2x}}}$

Solution: Let $u = e^x \ \rightarrow \ du = e^x dx \ \rightarrow \ \dfrac{du}{u} = dx$:

$$I = \int \frac{3}{u^2\sqrt{1+u^2}}du$$

$$u = \tan t, \quad du = \sec^2 t dt, \quad \sqrt{1+u^2} = \sec t$$

$$I = 3\int \frac{\sec^2 t\,dt}{\tan^2 t\sec t} = 3\int \frac{\sec t\,dt}{\tan^2 t} = 3\int \frac{1/\cos t}{\sin^2 t/\cos^2 t}\,dt = 3\int \frac{\cos t}{\sin^2 t}\,dt$$

Using another substitution $s = \sin t$

$$I = \frac{-3}{\sin t} + C = -3\frac{\sqrt{1+u^2}}{u} + C = \boxed{-\frac{3\sqrt{1+e^{2x}}}{e^x} + C}$$

(6) (**5 marks**) Find the area bounded by $x = 3 - y^2$ and $x = 1 - y$.

Solution:

Intersection:

$$3 - y^2 = 1 - y \Rightarrow y^2 - y - 2 = 0 \Rightarrow y = 2, -1$$

Area:

$$A = \int_{-1}^{2}[(3-y^2)-(1-y)]dy = \int_{-1}^{2}(2+y-y^2)dy$$

$$= \left[2y + \frac{y^2}{2} - \frac{y^3}{3}\right]_{-1}^{2} = \boxed{\frac{9}{2}}$$

(7) (**5 marks**) Solve:

$$y' \sec^2 x - 2e^{-y} \sin^3 x = 0$$

Solution:

$$\frac{dy}{dx} \sec^2 x = 2e^{-y} \sin^3 x$$

Separate:

$$e^y \, dy = 2 \sin^3 x \cos^2 x \, dx$$

Integrate:

$$e^y = \int 2 \sin^3 x \cos^2 x \, dx$$

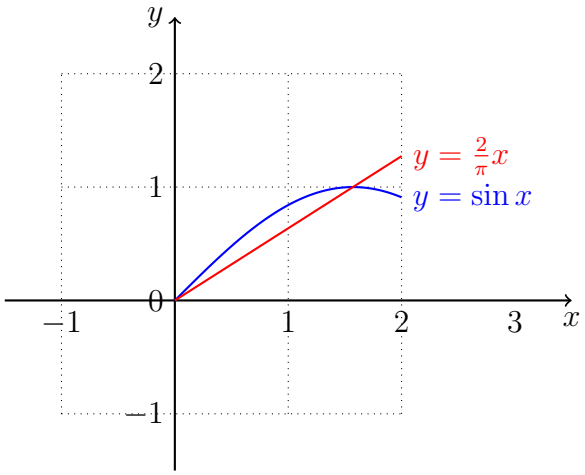
Let $u = \cos x$:

$$e^y = -2 \int (1 - u^2) u^2 \, du = -2 \int (u^2 - u^4) \, du = -\frac{2}{3} \cos^3 x + \frac{2}{5} \cos^5 x + C$$

$$\boxed{y = \ln \left(-\frac{2}{3} \cos^3 x + \frac{2}{5} \cos^5 x + C \right)}$$

(8) (**2 marks**) Sketch the region enclosed by $y = \sin x$ and $y = \frac{2}{\pi} x$ where $x > 0$ in the given coordinate system.

Solution:



(a) (**3 marks**) Set up (do not compute) the volume of the solid object obtained by rotating the region about the **x-axis.**

$$\boxed{V = \pi \int_0^{\frac{\pi}{2}} \left[(\sin x)^2 - \left(\frac{2}{\pi} x \right)^2 \right] dx}$$

(b) (**3 marks**) Set up (do not compute) the volume of the solid object obtained by rotating the region about the **y-axis.**

$$\boxed{V = 2\pi \int_0^{\frac{\pi}{2}} x \left(\sin x - \frac{2}{\pi} x \right) dx}$$

(9) (**10 marks**)

(a) $\displaystyle\int_{1/4}^{\infty} \frac{\arctan(4x)}{1+16x^2}dx$

Solution:

Let $u = \arctan(4x) \quad du = \dfrac{4}{1+16x^2}\,dx \;\rightarrow\; \displaystyle\int \frac{\arctan(4x)}{1+16x^2}dx = \frac{1}{8}(\arctan(4x))^2 + C$

$$I = \lim_{b\to\infty}\left[\frac{1}{8}(\arctan(4x))^2\right]_{1/4}^{b} = \frac{1}{8}\left[\frac{\pi^2}{4} - \frac{\pi^2}{16}\right] = \boxed{\frac{3\pi^2}{128}}$$

Convergent.

(b) $\displaystyle\int_0^2 (4x-3)\ln x dx$

Solution:

Since $\ln x \to -\infty$ as $x \to 0^+$,

$$\int_0^2 (4x-3)\ln x\,dx = \lim_{a\to 0^+}\int_a^2 (4x-3)\ln x\,dx.$$

We use integration by parts. Let

$$u = \ln x, \quad dv = (4x-3)\,dx.$$

$$du = \frac{1}{x}dx, \quad v = 2x^2 - 3x.$$

Thus,

$$\int (4x-3)\ln x\,dx = \left(2x^2-3x\right)\ln x - \int (2x-3)\;dx = \left(2x^2-3x\right)\ln x - \left(x^2-3x\right) + c.$$

Now evaluate the definite integral:

$$I = \lim_{a\to 0^+}\int_a^2 (4x-3)\ln x\,dx = \lim_{a\to 0^+}\left[\left(2x^2-3x\right)\ln x - \left(x^2-3x\right)\right]_a^2.$$

At $x = 2$:

$$2\ln 2 + 2.$$

At $x = a$:

$$\left(2a^2-3a\right)\ln a - \left(a^2-3a\right).$$

Now take the limit as $a \to 0^+$:

- $(2a^2-3a)\ln a \to 0$ (Using L'Hopital's Rule)
- $(a^2-3a) \to 0$

Therefore,

$$\boxed{\int_0^2 (4x-3)\ln x\,dx = 2 + \ln 4}$$

Convergent.

(10) (**9 marks**) Determine whether the series converges or diverges. Justify your answer and state the test/technique that you are applying.

(a) $\ln\left(\frac{1}{2}\right) + \ln\left(\frac{2}{3}\right) + \ln\left(\frac{3}{4}\right) + \cdots$

Solution:

$$S_n = \ln\frac{1}{2} + \ln\frac{2}{3} + \cdots + \ln\frac{n}{n+1}$$

$$S_n = \ln 1 - \ln 2 + \ln 2 - \ln 3 + \cdots + \ln n - \ln(n+1) = \ln 1 - \ln(n+1) = \ln\frac{1}{n+1}$$

$$\lim_{n\to\infty} S_n = -\infty$$

Telescoping Series, Divergent

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n!\ \tan(\frac{1}{n})}{(n-1)!}$

Solution:

$$\lim_{n\to\infty} \frac{n!\ \tan(\frac{1}{n})}{(n-1)!} = \lim_{n\to\infty} \frac{\tan(\frac{1}{n})}{(\frac{1}{n})} = 1$$

Test for Divergence, Divergent

(c) $\displaystyle\sum_{n=1}^{\infty} 2n\ e^{-n}$

Solution: Let

$$a_n = 2ne^{-n}.$$

We apply the Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{2(n+1)e^{-(n+1)}}{2ne^{-n}}.$$

Simplify:

$$= \lim_{n\to\infty} \frac{n+1}{n}\cdot e^{-1} = \left(\lim_{n\to\infty}\frac{n+1}{n}\right)\cdot\frac{1}{e} = 1\cdot\frac{1}{e} = \frac{1}{e}.$$

Since

$$\frac{1}{e} < 1,$$

the series converges absolutely by the Ratio Test.

Ratio Test, Absolutely convergent.

(11) **(3 marks)** What is the value of r if $\sum_{n=1}^{\infty} 2(r)^n = \frac{-4}{5}$.

Solution:

$$\frac{2r}{1-r} = -\frac{4}{5} \Rightarrow 10r = -4 + 4r \Rightarrow 6r = -4$$

$$\boxed{r = -\frac{2}{3}}$$

(12) (**10 marks**) Determine whether the series is absolutely convergent, conditionally convergent or divergent. Justify your answer and state the test that you are using.

(a) $\sum_{n=5}^{\infty} \frac{(-1)^n(3n+1)}{\sqrt{81n^4 - 18n^2 + 1}}$

Solution:
Note that $\sqrt{81n^4 - 18n^2 + 1} = \sqrt{(9n^2-1)^2} = 9n^2 - 1$.

Thus,

$$\frac{3n+1}{9n^2-1} = \frac{3n+1}{(3n-1)(3n+1)} = \frac{1}{3n-1}.$$

So the series becomes

$$\sum_{n=5}^{\infty} \frac{(-1)^n}{3n-1}.$$

Test for **absolute convergence:**

Since $\frac{1}{3n-1} > \frac{1}{n}$, and $\sum \frac{1}{n}$ is divergent, $\sum \frac{1}{3n-1}$ is divergent by the **Comparison Test.**
Hence, the series is not absolutely convergent.

Alternating Series Test:
Let $b_n = \frac{1}{3n-1}$. Then $b_n > 0$, decreasing, and $b_n \to 0$.
Thus the series converges.

$$\boxed{\text{The series is conditionally convergent.}}$$

(b) $\sum_{n=4}^{\infty} \frac{(-1)^n \sin(\frac{\pi}{n})}{1+2^n}$

Solution:

Test for **absolute convergence**:

$$\sum_{n=4}^{\infty} \left| \frac{(-1)^n \sin\left(\frac{\pi}{n}\right)}{1+2^n} \right| = \sum_{n=4}^{\infty} \frac{\sin\left(\frac{\pi}{n}\right)}{1+2^n}.$$

Since

$$0 \leq \frac{\sin(\pi/n)}{1+2^n} \leq \frac{1}{1+2^n} < \frac{1}{2^n}$$

and $\sum \frac{1}{2^n}$ converges (a geometric series with $|r| < 1$), we conclude by the Comparison Test that the series converges absolutely.

$$\boxed{\text{The series is absolutely convergent.}}$$

(13) (**6 marks**) Find the radius of convergence and the interval of convergence of the power series:

$$\sum_{n=2026}^{\infty} \frac{2^n(x-7)^n}{\sqrt{n-17}}$$

Solution;

Let

$$a_n = \frac{2^n(x-7)^n}{\sqrt{n-17}}.$$

Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{2^{n+1}|x-7|^{n+1}}{\sqrt{n-16}} \cdot \frac{\sqrt{n-17}}{2^n|x-7|^n}.$$

Simplify:

$$= \lim_{n\to\infty} 2|x-7| \cdot \sqrt{\frac{n-17}{n-16}}.$$

Since

$$\sqrt{\frac{n-17}{n-16}} \to 1,$$

we obtain

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 2|x-7|.$$

For convergence:

$$2|x-7| < 1 \quad \Rightarrow \quad |x-7| < \frac{1}{2}.$$

Thus, the radius of convergence is

$$\boxed{R = \frac{1}{2}}.$$

Interval of convergence:

$$7 - \frac{1}{2} < x < 7 + \frac{1}{2} \quad \Rightarrow \quad \frac{13}{2} < x < \frac{15}{2}.$$

Check endpoints:

$\underline{x = \frac{13}{2}}$:

$$x - 7 = -\frac{1}{2} \Rightarrow \sum \frac{(-1)^n}{\sqrt{n-17}}.$$

This is an alternating series with terms decreasing to 0, so it converges by the Alternating Series Test.

$\underline{x = \frac{15}{2}}$:

$$x - 7 = \frac{1}{2} \Rightarrow \sum \frac{1}{\sqrt{n-17}}.$$

This behaves like $\sum \frac{1}{\sqrt{n}}$, which diverges (p-series with $p = \frac{1}{2}$).

Final answer:

$$\boxed{R = \frac{1}{2}, \quad \left[\frac{13}{2}, \frac{15}{2}\right)}$$

(14) (**6 marks**) Use the derivative method to find the MacLaurin series of $\cos x$. Then use your answer to evaluate $\int \cos(x^2)\,dx$.

Solution:

Maclaurin:

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\cos(x^2) = \sum \frac{(-1)^n x^{4n}}{(2n)!}$$

Integrate:

$$\int \cos(x^2)dx = C + \sum \frac{(-1)^n x^{4n+1}}{(4n+1)(2n)!}$$

$$\boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{(4n+1)(2n)!} + C}$$