201-SN4-RE: Linear Algebra and Vector Geometry

Winter 2026 – Final Examination

Final Answers Only

1. Use Cramer's Rule to solve for x_1 **ONLY.** No other method is acceptable!

$$\begin{cases} x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 = 14, \\ 2x_2 + 3x_3 + 4x_4 + 5x_5 = 14, \\ 3x_3 + 4x_4 + 5x_5 = 12, \\ 4x_4 + 5x_5 = 9, \\ 5x_5 = 5. \end{cases}$$

Final answer:

$$x_1 = 0.$$

2. Determine the values of k for which the linear system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 8 & 4 & k^2 \\ 2k & k^2 & k+2 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ k+2 \\ 2 \end{bmatrix},$$

has

(a) Exactly one solution.

(b) No solutions.

(c) Infinitely many solutions.

Final answers:

(a) $k \neq -2, 0, 1, 2$.

(b) $k = -2, 0, 1$.

(c) $k = 2$.

3.

(a) Solve for the matrix X in

$$X(AX - X + 2I)^{-1} = A,$$

where A and $I + A - A^2$ are invertible.

(b) Find X from part (a) when

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}.$$

Final answers:

(a)
$$X = 2(I + A - A^2)^{-1}A.$$

(b)
$$X = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 12 \\ 0 & 0 & 2 \end{pmatrix}.$$

4. Let
$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$

(a) Find A^{-1}.

(b) Solve the system
$$\begin{cases} x + y = 1, \\ y + z = -2, \\ 2x + y + z = 3. \end{cases}$$

Final answers:

(a)
$$A^{-1} = \begin{pmatrix} 0 & -\frac{1}{2} & \frac{1}{2} \\ 1 & \frac{1}{2} & -\frac{1}{2} \\ -1 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

(b)
$$(x, y, z) = \left(\tfrac{5}{2}, -\tfrac{3}{2}, -\tfrac{1}{2}\right).$$

5. Let A and B be two 4×4 matrices such that
$$\det(A) = \begin{vmatrix} 1 & 1 & 0 & 0 \\ a + 2b & c + b & c & b \\ d + 2e & f + e & f & e \\ g + 2h & i + h & i & h \end{vmatrix} = 12 \quad \text{and} \quad \det(\operatorname{adj}(B)) = 8.$$

(a) Find $\begin{vmatrix} 13a & 13d & 13g \\ b & e & h \\ c & f & i \end{vmatrix}$.

(b) Find $\det(C)$ where $A^2CB^T = B + 2\operatorname{adj}(B^{-1})$.

Final answers:

(a)

$$156.$$

(b)

$$\det(C) = \tfrac{1}{9}.$$

6. Consider the points $A(1,1,0)$, $B(2,3,0)$, and $C(1+\lambda,3,0)$.

(a) Find the value(s) of λ, if any, for which $\{\vec{AB}, \vec{AC}\}$ is linearly dependent.

(b) Find the value(s) of λ, if any, for which the area of the triangle ABC is 2026.

(c) Find the distance between the point C and the line which passes through A and B when the area of the triangle ABC is 2026.

(d) Find the parametric equations of the line which passes through $P(1,0,2)$ and is parallel to $\vec{AB}$.

Final answers:

(a)

$$\lambda = 1.$$

(b)

$$\lambda = 2027 \text{ or } \lambda = -2025.$$

(c)

$$\frac{4052}{\sqrt{5}} = \frac{4052\sqrt{5}}{5}.$$

(d)

$$x = 1 + t, \quad y = 2t, \quad z = 2.$$

7. If $\vec{x} \cdot \vec{y} = 4$, $(\vec{x} - \vec{y}) \cdot (\vec{x} + \vec{y}) = -5$, and $(\vec{x} + \vec{y}) \cdot (\vec{x} + \vec{y}) = 21$, then find $\|\vec{x}\|$ and $\|\vec{y}\|$ and the cosine of the angle between $\vec{x}$ and $\vec{y}$.

Final answer:

$$\|\vec{x}\| = 2, \quad \|\vec{y}\| = 3, \quad \cos\theta = \frac{2}{3}.$$

8. Let P, Q, and X be $n \times n$ matrices. Assume that $AX = \mathbf{0}$ implies $A = \mathbf{0}$. Prove that if $PX = QX$ then $P = Q$.

Final answer:

$$P = Q.$$

9. Find all point(s), if any, which are at a distance of 2 of both $A(1,2,2)$ and $\mathcal{P} : x+2y+2z = -3$.

Final answer:

$$\left(\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{2}{3}\right).$$

10. Find the general equation of the plane containing the line

$$\mathcal{L} : (x, y, z) = (1, 2, 3) + t(2, 0, -4), \quad t \in \mathbb{R},$$

and perpendicular to the plane $\mathcal{P} : 2x + 3y + 6z = -1$.

Final answer:

$$6x - 10y + 3z + 5 = 0.$$

11. Consider the lines

$$\mathcal{L}_1 : \begin{cases} x = kt_1 + 1 \\ y = t_1 \\ z = 3t_1 + 1 \end{cases}, \ t_1 \in \mathbb{R} \qquad \text{and} \qquad \mathcal{L}_2 : \begin{cases} x = t_2 + 3 \\ y = t_2 + 1 \\ z = 3t_2 + 4 \end{cases}, \ t_2 \in \mathbb{R}.$$

(a) Determine the values of k, if any, for which the two lines are not skew.

(b) If $k = 2$, find the point(s) on each line which are closest to the other.

Final answers:

(a)

$$\text{All } k \in \mathbb{R}.$$

(b)

$$(3, 1, 4) \text{ on } \mathcal{L}_1 \text{ and } (3, 1, 4) \text{ on } \mathcal{L}_2.$$

12. Find the distance between the following two lines

$$\mathcal{L}_1 : \begin{cases} x = t_1 + 1 \\ y = t_1 \\ z = 3t_1 + 1 \end{cases}, \ t_1 \in \mathbb{R} \qquad \text{and} \qquad \mathcal{L}_2 : \begin{cases} x = t_2 + 3 \\ y = t_2 + 1 \\ z = 3t_2 + 4 \end{cases}, \ t_2 \in \mathbb{R}.$$

Final answer:

$$\sqrt{\frac{10}{11}}.$$

13. Suppose that both $\vec{u}$ and $\vec{v}$ are orthogonal to $\vec{w}$, that $\vec{u}$ is a unit vector, and that $\vec{v}$ is oppositely directed to $\vec{u}$ with twice its magnitude. Find the value of k such that $\text{proj}_{\vec{u}+\vec{v}}(\vec{w} + \vec{u}) = k\vec{v}$.

Final answer:

$$k = -\frac{1}{2}.$$

14. Let $\vec{u}$, $\vec{v}$, $\vec{w}$ be linearly independent vectors. Find λ so that the vectors

$$\vec{u} + \vec{v} + \vec{w}, \qquad \vec{u} + 2\vec{v} + \lambda\vec{w}, \qquad -\vec{u} + \vec{v} + \lambda\vec{w}$$

are linearly dependent.

Final answer:

$$\lambda = 3.$$

15. Let

$$W = \{(x, y, z) \in \mathbb{R}^3 \mid x + y - z = 0\}.$$

(a) Find a basis and the dimension of W.

(b) Express $(6, 7, 13)$ relative to the basis found in part (a).

Final answers:

(a) A basis is

$$\{(1, 0, 1), (0, 1, 1)\},$$

and

$$\dim(W) = 2.$$

(b) Relative coordinates:

$$(6, 7).$$

16. Determine whether each statement is true or false. If the statement is false provide a counterexample. If the statement is true provide a proof.

(a) Given A an invertible matrix. Then $I + A^{-1}$ is invertible if and only if $A + I$ is invertible.

(b) Any set of vectors that span $\mathbb{R}^3$ must be linearly independent.

(c) If A^k is equal to the zero matrix for some positive integer k, then the system $A\mathbf{x} = \mathbf{b}$ has a unique solution.

Final answers:

(a) True.

(b) False.

(c) False.