Dawson College

Mathematics Department

FINAL EXAMINATION

MULTIVARIABLE CALCULUS

201-SNC-DW, section 1

December 11, 2025 2:00pm – 5:00pm

Student Name: (Solutions)

Student ID #: _____

Instructor: A. Hariton

Time: 3 hours

Instructions:

Print your name and student ID number in the space provided above.

All questions are to be answered directly on the examination paper in the space provided. Show your complete work and give explanations.

A Sharp EL-531** calculator is permitted.

This examination consists of 18 questions. Please ensure that you have a complete examination.

This examination must be returned intact.

1. (5 marks)

Find the points on the parametric curve $x = t^3 + 3t^2$, $y = t^3 - 3t$ where the curve has a horizontal tangent and the points where the curve has a vertical tangent. What happens at the point $(4, -2)$?

$\frac{dx}{dt} = 3t^2 + 6t = 3t(t+2)$, so $\frac{dx}{dt} = 0$ if $t = 0$ or $t = -2$

$\frac{dy}{dt} = 3t^2 - 3 = 3(t^2-1) = 3(t+1)(t-1)$, so $\frac{dy}{dt} = 0$ if $t = \pm 1$

The curve has a horizontal tangent where $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$, at $t = \pm 1$. At $t = 1$, $(x, y) = (1+3, 1-3) = (4, -2)$

At $t = -1$, $(x, y) = (-1+3, -1+3) = (2, 2)$

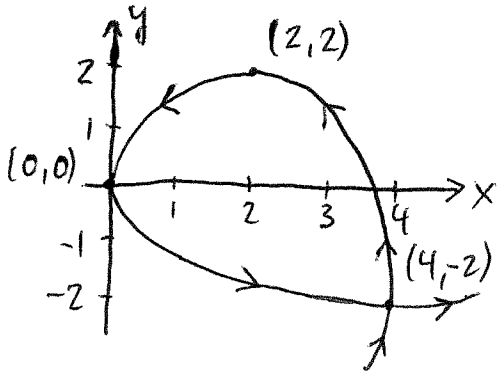
So the curve has a horizontal tangent at $\boxed{(4,-2) \text{ and } (2,2)}$

The curve has a vertical tangent where $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$, at $t = 0$ and $t = -2$. At $t = 0$, $(x, y) = (0, 0)$

At $t = -2$, $(x, y) = (-8+12, -8+6) = (4, -2)$

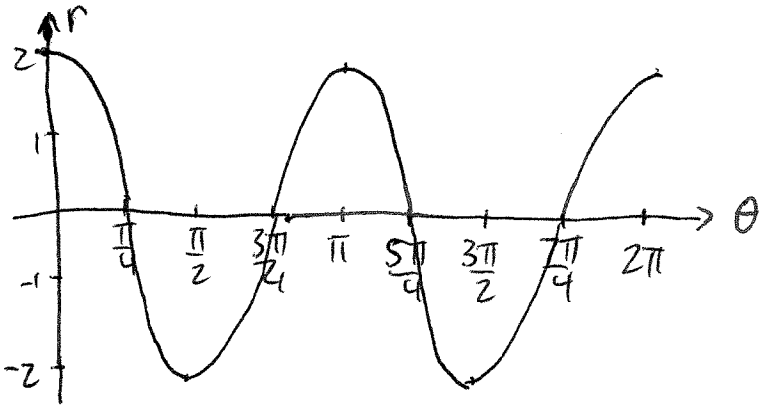
So the curve has a vertical tangent at $\boxed{(0,0) \text{ and } (4,-2)}$

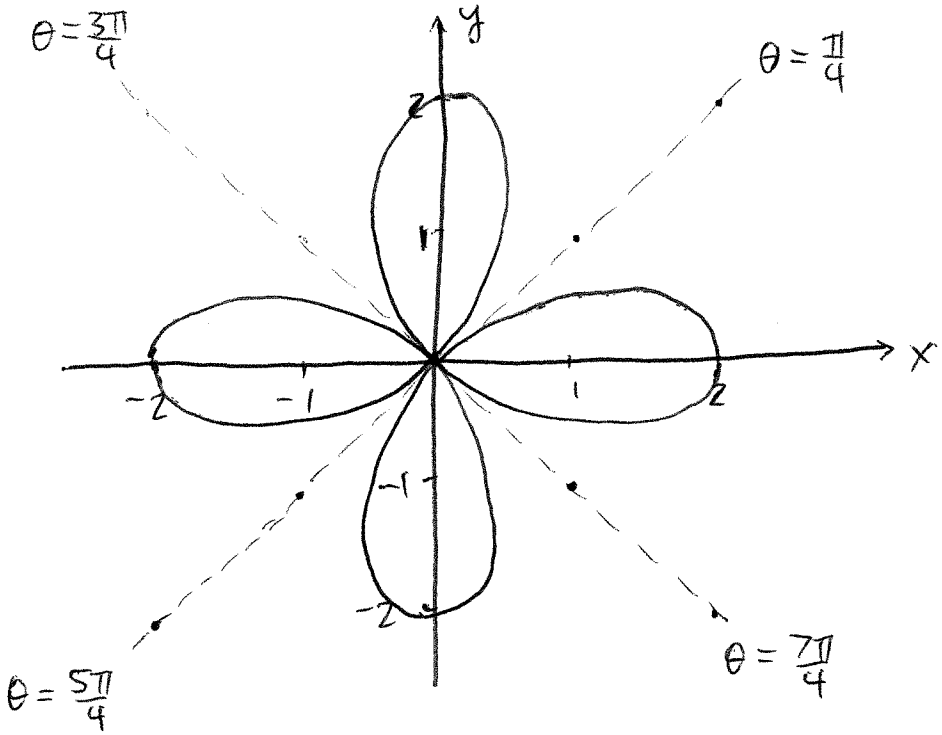
At the point $(4, -2)$, the curve crosses itself, so it has two tangents, one horizontal and one vertical.



2. (5 marks)

Sketch the polar curve $r = 2\cos(2\theta)$





3. (6 marks)

Consider the vector function $\vec{r}(t) = \langle \sin(\pi t) , \frac{e^t - 1}{t} , \ln(4t + 3) \rangle$

(a) Find $\lim_{t \to 0} \vec{r}(t)$

$$\lim_{t\to 0} \vec{r}(t) = \left\langle \lim_{t\to 0} \sin(\pi t), \lim_{t\to 0} \frac{e^t - 1}{t}, \lim_{t\to 0} \ln(4t+3) \right\rangle$$

$$= \left\langle \sin(0), \lim_{t\to 0} \frac{e^t}{1}, \ln(0+3) \right\rangle$$

$$= \boxed{\langle 0, 1, \ln 3 \rangle}$$

(b) Find the parametric equations of the tangent line to the space curve with vector function $\vec{r}(t)$ at the point $(0, e - 1, \ln 7)$

$$\vec{r}\,'(t) = \left\langle \pi \cos(\pi t), \frac{te^t - e^t + 1}{t^2}, \frac{4}{4t+3} \right\rangle$$

At point $(0, e-1, \ln 7)$, $\ln(4t+3) = \ln 7$, so $4t+3 = 7$, $t = 1$

$$\vec{r}\,'(1) = \left\langle \pi \cos(\pi), \frac{e - e + 1}{(1)}, \frac{4}{4+3} \right\rangle = \left\langle -\pi, 1, \frac{4}{7} \right\rangle$$

So the parametric equations of the tangent line are:

$$\boxed{x = -\pi t, \quad y = e - 1 + t, \quad z = \ln 7 + \frac{4}{7} t}$$

4. (6 marks)

Parametrize the space curve

$$\vec{r}(t) = \langle e^{2t}\sin t,\ e^{2t}\cos t,\ 2e^{2t}\rangle,\quad t \geq 0$$

with respect to arc length measured from the point $(0, 1, 2)$ in the direction of increasing t

$\vec{r}(t) = (0,1,2)$ when $2e^{2t} = 2$, so $e^{2t} = 1$, $t = 0$

$\vec{r}'(t) = \langle 2e^{2t}\sin t + e^{2t}\cos t,\ 2e^{2t}\cos t - e^{2t}\sin t,\ 4e^{2t}\rangle$

$$\|\vec{r}'(t)\| = \sqrt{(2e^{2t}\sin t + e^{2t}\cos t)^2 + (2e^{2t}\cos t - e^{2t}\sin t)^2 + (4e^{2t})^2}$$

$$= \sqrt{4e^{4t}\sin^2 t + 4e^{4t}\sin t\cos t + e^{4t}\cos^2 t + 4e^{4t}\cos^2 t - 4e^{4t}\sin t\cos t + e^{4t}\sin^2 t + 16e^{4t}}$$

$$= \sqrt{4e^{4t} + e^{4t} + 16e^{4t}} = \sqrt{21\,e^{4t}} = \sqrt{21}\,e^{2t}$$

$$s = \int_0^t \|\vec{r}'(u)\|\,du = \int_0^t \sqrt{21}\,e^{2u} = \frac{\sqrt{21}}{2}e^{2u}\Big|_0^t$$

$$= \frac{\sqrt{21}}{2}e^{2t} - \frac{\sqrt{21}}{2}$$

$$\frac{2s}{\sqrt{21}} = e^{2t} - 1,\quad \text{so}\quad e^{2t} = \frac{2s+\sqrt{21}}{\sqrt{21}},\quad t = \frac{1}{2}\ln\left(\frac{2s+\sqrt{21}}{\sqrt{21}}\right)$$

$$\vec{r}(t(s)) = \left\langle \left(\frac{2s+\sqrt{21}}{\sqrt{21}}\right)\sin\left(\frac{1}{2}\ln\left(\frac{2s+\sqrt{21}}{\sqrt{21}}\right)\right),\ \left(\frac{2s+\sqrt{21}}{\sqrt{21}}\right)\cos\left(\frac{1}{2}\ln\left(\frac{2s+\sqrt{21}}{\sqrt{21}}\right)\right),\ \frac{4s+2\sqrt{21}}{\sqrt{21}}\right\rangle$$

5. (8 marks)

Consider the space curve $\vec{r}(t) = \langle \cos(2t)\ ,\ \sin(2t)\ ,\ 4t \rangle$

(a) Find the unit tangent vector $\vec{T}(t)$, the unit normal vector $\vec{N}(t)$ and the binormal vector $\vec{B}(t)$ to the curve $\vec{r}(t)$

$$\vec{r}\,'(t) = \langle -2\sin(2t),\ 2\cos(2t),\ 4 \rangle$$

$$\|\vec{r}\,'(t)\| = \sqrt{4\sin^2(2t) + 4\cos^2(2t) + 16} = \sqrt{4+16} = \sqrt{20} = 2\sqrt{5}$$

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \frac{1}{2\sqrt{5}}\langle -2\sin(2t),\ 2\cos(2t),\ 4 \rangle = \boxed{\left\langle -\frac{\sin(2t)}{\sqrt{5}},\ \frac{\cos(2t)}{\sqrt{5}},\ \frac{2}{\sqrt{5}} \right\rangle}$$

$$\vec{T}\,'(t) = \left\langle -\frac{2\cos(2t)}{\sqrt{5}},\ -\frac{2\sin(2t)}{\sqrt{5}},\ 0 \right\rangle$$

$$\|\vec{T}\,'(t)\| = \sqrt{\frac{4\cos^2(2t)}{5} + \frac{4\sin^2(2t)}{5} + 0} = \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}}$$

$$\vec{N}(t) = \frac{\vec{T}\,'(t)}{\|\vec{T}\,'(t)\|} = \frac{\sqrt{5}}{2}\left\langle -\frac{2\cos(2t)}{\sqrt{5}},\ -\frac{2\sin(2t)}{\sqrt{5}},\ 0 \right\rangle = \boxed{\langle -\cos(2t),\ -\sin(2t),\ 0 \rangle}$$

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\frac{\sin(2t)}{\sqrt{5}} & \frac{\cos(2t)}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\cos(2t) & -\sin(2t) & 0 \end{vmatrix} = \boxed{\left\langle \frac{2\sin(2t)}{\sqrt{5}},\ -\frac{2\cos(2t)}{\sqrt{5}},\ \frac{1}{\sqrt{5}} \right\rangle}$$

(b) Find the equation of the osculating plane to the curve $\vec{r}(t)$ at the point on the curve corresponding to the value $t = \frac{\pi}{2}$

$$\vec{B}\left(\tfrac{\pi}{2}\right) = \left\langle \frac{2\sin(\pi)}{\sqrt{5}},\ \frac{-2\cos(\pi)}{\sqrt{5}},\ \frac{1}{\sqrt{5}} \right\rangle = \left\langle 0,\ \frac{2}{\sqrt{5}},\ \frac{1}{\sqrt{5}} \right\rangle$$

At $t = \frac{\pi}{2}$, $\vec{r}\left(\tfrac{\pi}{2}\right) = \langle \cos(\pi),\ \sin(\pi),\ 2\pi \rangle = \langle -1,\ 0,\ 2\pi \rangle$

The equation of the osculating plane is:

$$(0)(x-(-1)) + \left(\frac{2}{\sqrt{5}}\right)(y-0) + \left(\frac{1}{\sqrt{5}}\right)(z-2\pi) = 0$$

$$\boxed{2y + z - 2\pi = 0}$$

6. (8 marks)

A particle has an acceleration of $\vec{a}(t) = \langle 2\ ,\ 6t\ ,\ 3t^2 \rangle$, an initial position of $\vec{r}(0) = \langle 3\ ,\ 1\ ,\ 0 \rangle$ and an initial velocity of $\vec{v}(0) = \langle -1\ ,\ 0\ ,\ 1 \rangle$

(a) Find the position function $\vec{r}(t)$ of the particle

$\vec{v}(t) = \langle 2t, 3t^2, t^3 \rangle + \langle C_1, C_2, C_3 \rangle$. $\vec{v}(0) = \langle -1, 0, 1 \rangle$, so

$\vec{v}(t) = \langle 2t-1, 3t^2, t^3+1 \rangle$

$\vec{r}(t) = \langle t^2-t, t^3, \frac{1}{4}t^4+t \rangle + \langle K_1, K_2, K_3 \rangle$. $\vec{r}(0) = \langle 3, 1, 0 \rangle$, so

$$\boxed{\vec{r}(t) = \langle t^2-t+3\ ,\ t^3+1\ ,\ \tfrac{1}{4}t^4+t \rangle}$$

(b) Find the curvature of the space curve defined by the position function of the particle at the point $\left(3,\ 2,\ \frac{5}{4}\right)$

$\vec{r}(t) = \langle 3, 2, \frac{5}{4} \rangle$ when $t^3+1 = 2$, $t^3 = 1$, $t = 1$

$\vec{r}\,'(1) = \vec{v}(1) = \langle 2(1)-1, 3(1)^2, (1)^3+1 \rangle = \langle 1, 3, 2 \rangle$

$\|\vec{r}\,'(1)\| = \sqrt{(1)^2+(3)^2+(2)^2} = \sqrt{14}$

$\vec{r}\,''(1) = \vec{a}(1) = \langle 2, 6(1), 3(1)^2 \rangle = \langle 2, 6, 3 \rangle$

$$\vec{r}\,'(1) \times \vec{r}\,''(1) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & 2 \\ 2 & 6 & 3 \end{vmatrix} = \langle -3, 1, 0 \rangle$$

$\|\vec{r}\,'(1) \times \vec{r}\,''(1)\| = \sqrt{(-3)^2+(1)^2+(0)^2} = \sqrt{10}$

Curvature: $K(1) = \dfrac{\|\vec{r}\,'(1) \times \vec{r}\,''(1)\|}{\|\vec{r}\,'(1)\|^3} = \dfrac{\sqrt{10}}{(\sqrt{14})^3} = \boxed{\dfrac{\sqrt{10}}{(14)^{3/2}}}$

7. (5 marks)

Find the limit

$$\lim_{(x,y)\to(0,0)} \frac{2xy^4}{3x^2+7y^8}$$

or show that the limit does not exist. Justify your answer.

As (x,y) approaches $(0,0)$ along the x-axis $(y=0)$:

$$\frac{2xy^4}{3x^2+7y^8} = \frac{0}{3x^2+0} = \frac{0}{3x^2} = 0, \text{ so } \lim \to 0$$

As (x,y) approaches $(0,0)$ along the curve $x=y^4$:

$$\frac{2xy^4}{3x^2+7y^8} = \frac{2(y^4)y^4}{3(y^4)^2+7y^8} = \frac{2y^8}{3y^8+7y^8} = \frac{2y^8}{10y^8} = \frac{1}{5}$$

so $\lim \to \frac{1}{5}$

Approaching $(0,0)$ along two different paths, we approach two different values, so $\lim_{(x,y)\to(0,0)} \frac{2xy^4}{3x^2+7y^8}$ does not exist.

8. (4 marks)

Find the equation of the tangent plane to the surface

$$z = \frac{3x}{y} + \frac{4y}{x}$$

at the point $(2, 3, 8)$.

$z_x = \frac{3}{y} - \frac{4y}{x^2}$, so $z_x(2,3) = \frac{3}{3} - \frac{4(3)}{(2)^2} = 1 - \frac{12}{4} = -2$

$z_y = -\frac{3x}{y^2} + \frac{4}{x}$, so $z_y(2,3) = -\frac{3(2)}{(3)^2} + \frac{4}{2} = -\frac{2}{3} + 2 = \frac{4}{3}$

The equation of the tangent plane is:

$z - 8 = (-2)(x-2) + (\frac{4}{3})(y-3)$

$z - 8 = -2x + 4 + \frac{4}{3}y - 4$

$2x - \frac{4}{3}y + z - 8 = 0$

$$\boxed{6x - 4y + 3z - 24 = 0}$$

9. (6 marks)

If $w = xe^{yz} - y\cos(x+z)$, where $x = r^2 s + 3r$, $y = r\ln s$ and $z = \arctan(rs)$, then find $\dfrac{\partial w}{\partial r}$ and $\dfrac{\partial w}{\partial s}$

(the answers can be given in terms of x, y, z, r and s)

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$= \left(e^{yz} + y\sin(x+z)\right)(2rs+3) + \left(xze^{yz} - \cos(x+z)\right)(\ln s) + \left(xye^{yz} + y\sin(x+z)\right)\left(\frac{s}{1+r^2s^2}\right)$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

$$= \left(e^{yz} + y\sin(x+z)\right)(r^2) + \left(xze^{yz} - \cos(x+z)\right)\left(\frac{r}{s}\right) + \left(xye^{yz} + y\sin(x+z)\right)\left(\frac{r}{1+r^2s^2}\right)$$

10. (7 marks)

Let $f(x,y,z) = x^2y + 2x\sqrt{3z-4y}$

(a) Find the directional derivative of f at the point $(1,2,3)$ in the direction of the vector $\vec{v} = \langle -2,6,3 \rangle$

$$\vec{\nabla} f(x,y,z) = \left\langle 2xy + 2\sqrt{3z-4y},\ x^2 - \frac{4x}{\sqrt{3z-4y}},\ \frac{3x}{\sqrt{3z-4y}} \right\rangle$$

$$\vec{\nabla} f(1,2,3) = \left\langle 2(1)(2) + 2\sqrt{3(3)-4(2)},\ (1)^2 - \frac{4(1)}{\sqrt{3(3)-4(2)}},\ \frac{3(1)}{\sqrt{3(3)-4(2)}} \right\rangle$$

$$= \left\langle 4+2,\ 1-\frac{4}{1},\ \frac{3}{1} \right\rangle = \langle 6,-3,3 \rangle$$

Unit vector in the direction of $\vec{v}$: $\vec{u} = \frac{1}{\sqrt{4+36+9}}\langle -2,6,3 \rangle$

$$= \left\langle -\frac{2}{7}, \frac{6}{7}, \frac{3}{7} \right\rangle$$

$$D_{\vec{u}} f(1,2,3) = \vec{\nabla} f(1,2,3) \cdot \vec{u} = \langle 6,-3,3 \rangle \cdot \left\langle -\frac{2}{7}, \frac{6}{7}, \frac{3}{7} \right\rangle$$

$$= -\frac{12}{7} - \frac{18}{7} + \frac{9}{7} = -\frac{21}{7} = \boxed{-3}$$

(b) Find the maximum rate of change of the function f at the point $(1,2,3)$

The maximum rate of change is:

$$\|\vec{\nabla} f(1,2,3)\| = \sqrt{(6)^2 + (-3)^2 + (3)^2} = \sqrt{54} = \boxed{3\sqrt{6}}$$

11. (5 marks)

Find the local maxima, local minima and saddle points of the function

$f(x,y) = 3x^2y + y^3 - 3y$

$f_x = 6xy$, $f_y = 3x^2 + 3y^2 - 3$.

If $f_x = 0$, then $6xy = 0$, so either $x=0$ or $y=0$

If $x=0$, then $f_y = 0$ gives $3y^2 - 3 = 0$, $y^2 = 1$, $y = \pm 1$

If $y=0$, then $f_y = 0$ gives $3x^2 - 3 = 0$, $x^2 = 1$, $x = \pm 1$

There are four critical points: $(0,1)$, $(0,-1)$, $(1,0)$, $(-1,0)$

$f_{xx} = 6y$, $f_{yy} = 6y$, $f_{xy} = 6x$

$D = f_{xx}f_{yy} - (f_{xy})^2 = (6y)^2 - (6x)^2 = 36(y^2 - x^2)$

Point $(0,1)$, $D = 36(1-0) = 36 > 0$, $f_{xx} = 6(1) = 6 > 0$,
so there is a local minimum at $(0,1)$, with value -2

Point $(0,-1)$, $D = 36(1-0) = 36 > 0$, $f_{xx} = 6(-1) = -6 < 0$
so there is a local maximum at $(0,-1)$, with value 2

Points $(1,0)$ and $(-1,0)$ $D = 36(0-1) = -36 < 0$, so
$(1,0)$ and $(-1,0)$ are saddle points.

12. (5 marks)

Find the maximum and minimum values of the function $f(x,y,z) = e^{xyz}$ subject to the constraint $x^2 + y^2 + 2z^2 = 6$

$f(x,y,z) = e^{xyz}$, $g(x,y,z) = x^2+y^2+2z^2 = 6$

$f_x = \lambda g_x$, so $\boxed{yze^{xyz} = 2x\lambda}$ (1)

$f_y = \lambda g_y$, so $\boxed{xze^{xyz} = 2y\lambda}$ (2)

$f_z = \lambda g_z$, so $\boxed{xye^{xyz} = 4z\lambda}$ (3)

$\boxed{x^2+y^2+2z^2 = 6}$ (4)

If $\lambda = 0$, then $yz = xz = xy = 0$, which means that either $x=y=z=0$ (impossible since it contradicts (4)) or only one of x, y and z is non-zero. If $x \neq 0$, $y=0$, $z=0$, then $x^2=6$, so $x = \pm\sqrt{6}$: $(\sqrt{6},0,0)$, $(-\sqrt{6},0,0)$. Likewise, if $y \neq 0$, we get $(0,\sqrt{6},0)$ and $(0,-\sqrt{6},0)$. Finally, if $z \neq 0$, we get $(0,0,\sqrt{3})$ and $(0,0,-\sqrt{3})$. In all cases, $f = e^0 = \boxed{1}$

If $\lambda \neq 0$, then (1), (2) and (3) give $2x^2\lambda = 2y^2\lambda = 4z^2\lambda$, so $x^2 = y^2 = 2z^2$. Then (4) gives $6z^2 = 6$, so $z^2 = 1$ so $x = \pm\sqrt{2}$, $y = \pm\sqrt{2}$, $z = \pm 1$. We have eight points $(\pm\sqrt{2}, \pm\sqrt{2}, \pm 1)$, for which $f = \boxed{e^2}$ or $f = \boxed{e^{-2}}$.

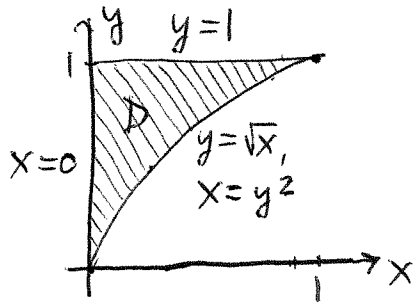
The maximum is e^2 and the minimum is e^{-2}

13. (5 marks)

Evaluate the iterated integral

$$\int_0^1 \int_{\sqrt{x}}^1 \cos(y^3)\ dy\ dx$$

$$\int_0^1 \int_{\sqrt{x}}^1 \cos(y^3)\,dy\,dx = \iint_D \cos(y^3)\,dA$$



where $D = \{(x,y) \mid 0 \le x \le 1,\ \sqrt{x} \le y \le 1\}$

However, D can also be written as:

$D = \{(x,y) \mid 0 \le y \le 1,\ 0 \le x \le y^2\}$, so

$$\int_0^1 \int_{\sqrt{x}}^1 \cos(y^3)\,dy\,dx = \int_0^1 \int_0^{y^2} \cos(y^3)\,dx\,dy$$

$$= \int_0^1 x\cos(y^3)\Big|_{x=0}^{x=y^2} dy = \int_0^1 y^2\cos(y^3)\,dy,$$

$u = y^3$
$du = 3y^2dy$
$\frac{1}{3}du = y^2dy$
$y=0 \to u=0$
$y=1 \to u=1$

$$= \int_0^1 \frac{1}{3}\cos u\,du = \frac{1}{3}\sin u\Big|_0^1$$

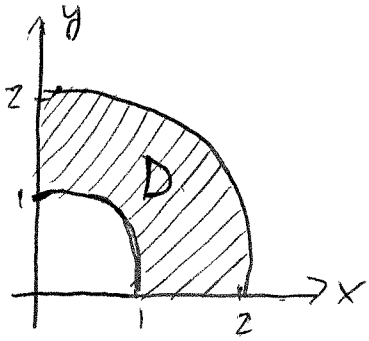
$$= \frac{1}{3}\sin(1) - \frac{1}{3}\sin(0) = \boxed{\frac{\sin(1)}{3}}$$

14. (5 marks)

Evaluate the double integral

$$\iint_D (3x + 2y)\, dA$$

where D is the region in the first quadrant $(x \geq 0,\ y \geq 0)$ bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$



In polar coordinates,

$D = \{(r,\theta) \mid 0 \leq \theta \leq \frac{\pi}{2},\ 1 \leq r \leq 2\}$, so

$$\iint_D (3x+2y)\,dA$$

$$= \int_0^{\frac{\pi}{2}} \int_1^2 \left(3(r\cos\theta) + 2(r\sin\theta)\right) r\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_1^2 \left(3r^2\cos\theta + 2r^2\sin\theta\right) dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(r^3\cos\theta + \frac{2}{3}r^3\sin\theta\right)\Big|_{r=1}^{r=2} d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(8\cos\theta + \frac{16}{3}\sin\theta - \cos\theta - \frac{2}{3}\sin\theta\right) d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(7\cos\theta + \frac{14}{3}\sin\theta\right) d\theta$$

$$= \left(7\sin\theta - \frac{14}{3}\cos\theta\right)\Big|_0^{\frac{\pi}{2}} = 7 - 0 - 0 + \frac{14}{3}$$

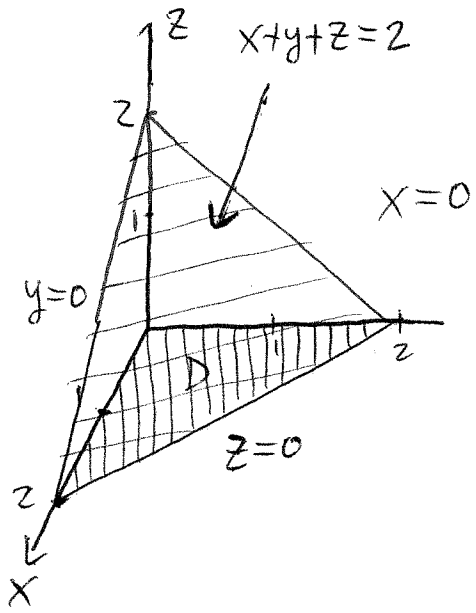
$$= \boxed{\frac{35}{3}}$$

15. (5 marks)

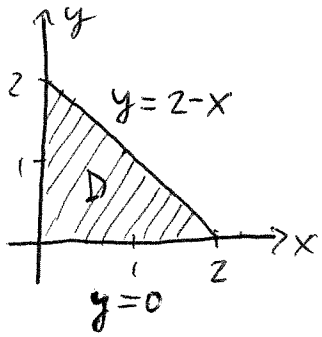
Find the volume of the tetrahedron bounded by the four planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 2$

The intersection of the plane $x+y+z=2$ with the xy-plane ($z=0$) is:

$x+y=2$, that is $y = 2-x$



The tetrahedron E is bounded below by $z=0$ and bounded above by $z = 2-x-y$, so the volume of the tetrahedron is:



$$V = \int_0^2 \int_0^{2-x} \int_0^{2-x-y} dz\, dy\, dx$$

$$= \int_0^2 \int_0^{2-x} (2-x-y)\, dy\, dx$$

$$= \int_0^2 \left(2y - xy - \tfrac{1}{2}y^2\right)\Big|_{y=0}^{y=2-x} dx$$

$$= \int_0^2 \left(2(2-x) - x(2-x) - \tfrac{1}{2}(2-x)^2\right) dx$$

$$= \int_0^2 \left(4 - 2x - 2x + x^2 - 2 + 2x - \tfrac{1}{2}x^2\right) dx$$

$$= \int_0^2 \left(\tfrac{1}{2}x^2 - 2x + 2\right) dx = \left(\tfrac{1}{6}x^3 - x^2 + 2x\right)\Big|_0^2 = \frac{8}{6} - 4 + 4$$

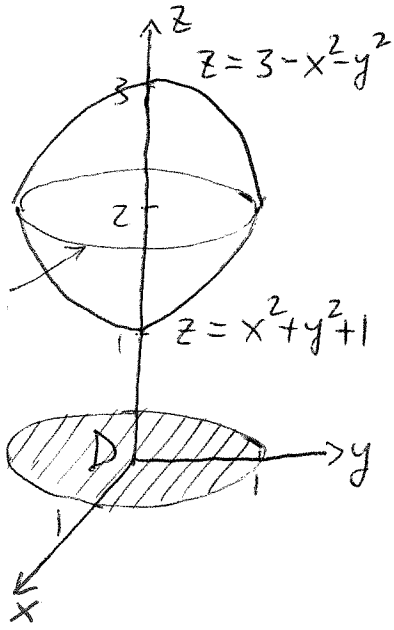
$$= \boxed{\frac{4}{3}}$$

16. (5 marks)

Use cylindrical coordinates to evaluate the triple integral

$$\iiint_E xy \, dV$$

where E is the solid that lies above the paraboloid $z = x^2 + y^2 + 1$ and below the paraboloid $z = 3 - x^2 - y^2$



The intersection of the two paraboloids is where

$x^2+y^2+1 = 3-x^2-y^2$, so $2x^2+2y^2 = 2$,

$\boxed{x^2+y^2=1}$ and $z = x^2+y^2+1 = 1+1 = 2$.

The curve $\boxed{x^2+y^2=1, \quad z=2}$ is the intersection

so D is the unit disk.

$E = \{(r,\theta,z) \mid 0 \le \theta \le 2\pi, \; 0 \le r \le 1, \; r^2+1 \le z \le 3-r^2\}$

$$\iiint_E xy \, dV = \int_0^{2\pi} \int_0^1 \int_{r^2+1}^{3-r^2} (r\cos\theta)(r\sin\theta)\, r \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^1 r^3 \cos\theta \sin\theta \left((3-r^2)-(r^2+1)\right) dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^1 (2r^3 - 2r^5) \cos\theta \sin\theta \, dr \, d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{2}r^4 - \frac{1}{3}r^6\right) \cos\theta \sin\theta \Big|_{r=0}^{r=1} d\theta$$

$$= \frac{1}{6}\int_0^{2\pi} \cos\theta \sin\theta \, d\theta = \frac{1}{6}\left(\frac{1}{2}\sin^2\theta\right)\Big|_0^{2\pi} = \frac{1}{12}(0-0) = \boxed{0}$$

17. (5 marks)

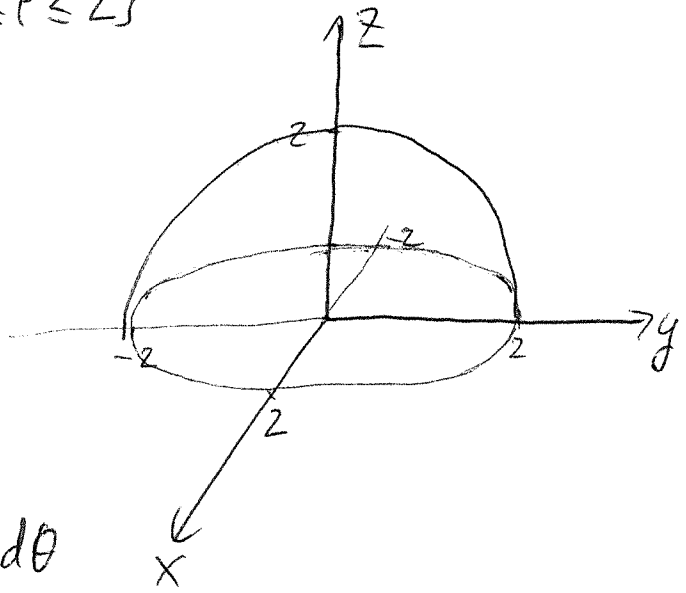
Use spherical coordinates to evaluate the triple integral

$$\iiint_E e^{(x^2+y^2+z^2)^{3/2}}\, dV$$

where $E = \{(x,y,z) \mid x^2+y^2+z^2 \le 4,\ \ z \ge 0\}$.

$E = \{(\rho,\theta,\phi) \mid 0 \le \theta \le 2\pi,\ 0 \le \phi \le \frac{\pi}{2},\ 0 \le \rho \le 2\}$

$e^{(x^2+y^2+z^2)^{3/2}} = e^{(\rho^2)^{3/2}} = e^{\rho^3}$



$$\iiint_E e^{(x^2+y^2+z^2)^{3/2}}\, dV$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{2} e^{\rho^3}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \frac{1}{3}e^{\rho^3}\sin\phi\,\Big|_{\rho=0}^{\rho=2}\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \frac{1}{3}(e^8-1)\sin\phi\, d\phi\, d\theta$$

$$= \frac{1}{3}(e^8-1)\int_0^{2\pi}(-\cos\phi)\,\Big|_{\phi=0}^{\phi=\frac{\pi}{2}}\, d\theta$$

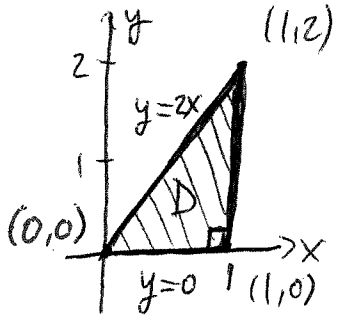
$$= \frac{1}{3}(e^8-1)\int_0^{2\pi}(-0+1)\, d\theta$$

$$= \boxed{\frac{2\pi}{3}(e^8-1)}$$

18. (5 marks)

Find the center of mass of a triangular lamina with vertices at $(0,0)$, $(1,0)$ and $(1,2)$ if the density function is $\rho(x,y) = 4xy$

$D = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le 2x\}$



$$m = \iint_D \rho(x,y)\,dA = \int_0^1 \int_0^{2x} 4xy\,dy\,dx$$

$$= \int_0^1 2xy^2 \Big|_{y=0}^{y=2x} dx = \int_0^1 8x^3\,dx$$

$$= 2x^4 \Big|_0^1 = 2 - 0 = \boxed{2}$$

$$\bar{x} = \frac{1}{m} \iint_D x\,\rho(x,y)\,dA = \frac{1}{2} \int_0^1 \int_0^{2x} 4x^2 y\,dy\,dx = \frac{1}{2} \int_0^1 2x^2 y^2 \Big|_{y=0}^{y=2x} dx$$

$$= \frac{1}{2} \int_0^1 8x^4\,dx = \frac{1}{2} \left(\frac{8}{5}x^5\right)\Big|_0^1 = \frac{8}{10} = \boxed{\frac{4}{5}}$$

$$\bar{y} = \frac{1}{m} \iint_D y\,\rho(x,y)\,dA = \frac{1}{2} \int_0^1 \int_0^{2x} 4xy^2\,dy\,dx = \frac{1}{2} \int_0^1 \frac{4}{3}xy^3 \Big|_{y=0}^{y=2x} dx$$

$$= \frac{1}{2} \int_0^1 \frac{32}{3}x^4\,dx = \frac{1}{2} \left(\frac{32}{15}x^5\right)\Big|_0^1 = \boxed{\frac{16}{15}}$$

The center of mass is at $\boxed{\left(\frac{4}{5}, \frac{16}{15}\right)}$