Polynomials Long Division

In algebra, **polynomial long division** is an algorithm for dividing a polynomial by another polynomial of the same or lower degree, a generalised version of the familiar arithmetic technique called long division. We illustrate the algorithm trough the examples below.

An Illustrative Example. Given the rational expression

$$\frac{x^3 - 10x^2 + 17}{x - 4},$$

divide $x^3 - 10x^2 + 17$ (the dividend or the numerator) by $x - 4$ (the divisor or the denominator).

1. We first set up the division: $x - 4 \quad \overline{|\ x^3 - 10x^2 + 0x + 17}$.

2. We now divide the leading term of the divided, namely x^3, by the leading term of the divisor, which is in this case x, (i.e., $x^3 \div x = x^2$) and then place the result above the bar:

$$\begin{array}{ll} & x^2 \\ x - 4 & \overline{|\ x^3 - 10x^2 + 0x + 17} \end{array}$$

3. Next we multiply the divisor by the result just obtained ($x^2(x - 4) = x^3 - 4x^2$) and write the result under the first two terms of the numerator:

$$\begin{array}{ll} & x^2 \\ x - 4 & \overline{|\ x^3 - 10x^2 + 0x + 17} \\ & \quad x^3 - 4x^2 \end{array}$$

4. Subtract then the product just obtained from the appropriate terms of the original numerator (being careful that subtracting something having a minus sign is equivalent to adding something having a plus sign), and write the result underneath ($(x^3 - 10x^2) - (x^3 - 4x^2) = -10x^2 + 4x^2 = -6x^2$). Then, "bring down" the next term from the numerator:

$$\begin{array}{ll} & x^2 \\ x - 4 & \overline{|\ x^3 - 10x^2 + 0x + 17} \\ & \quad \underline{x^3 - 4x^2} \\ & \qquad - 6x^2 + 0x + 17 \end{array}$$

5. Repeat the previous three steps, except this time use the two terms that have just been written as the numerator:

$$\begin{array}{ll} & x^2 - 6x \\ x - 4 & \overline{|\ x^3 \ - 10x^2 + 0x + 17} \\ & \quad \underline{x^3 - 4x^2} \\ & \qquad - 6x^2 + 0x + 17 \\ & \qquad \underline{-6x^2 + 24x} \\ & \qquad\qquad\quad - 24x + 17 \end{array}$$

6. Repeat 5. This time, there is nothing to "pull down":

$$
\begin{array}{r|l}
 & \quad x^2 - 6x - 24 \\
\hline
x-4 & x^3 - 10x^2 + 0x + 17 \\
 & \underline{x^3 - 4x^2} \\
 & \qquad - 6x^2 + 0x + 17 \\
 & \qquad \underline{-6x^2 + 24x} \\
 & \qquad\qquad - 24x + 17 \\
 & \qquad\qquad \underline{-24x + 96} \\
 & \qquad\qquad\qquad -79
\end{array}
$$

It is time to stop! The polynomial above the bar is the quotient, and the number left over (-79) is the remainder.

$$\frac{\overbrace{x^3 - 10x^2 + 17}^{\text{dividend}}}{\underbrace{x-4}_{\text{divisor}}} = \underbrace{x^2 - 6x - 24}_{\text{quotient}} + \frac{\overbrace{-79}^{\text{remainder}}}{x-4}.$$

Verification: It is recommended that each time we divide by L.D., we verify our computation according to the following:

$$\text{numrator} = \text{denominator} \cdot \text{quotient} + \text{remainder}.$$

For instance, let us do the verification of the example above:

$$
\begin{aligned}
(x-4)(x^2 - 6x - 24) + (-79) &= x^3 - 6x^2 - 24x - 4x^2 + 24x + 96 - 79 \\
&= x^3 - 10x^2 + 17.
\end{aligned}
$$

Another Example Given $\dfrac{x^4 + 4x^3 - 5x^2 - 12x + 6}{x^2 - 3}$, use L.D. to find the quotient and the remainder.

Here are the computations:

$$
\begin{array}{r|l}
 & \quad x^2 - 4x - 2 \longleftarrow \text{quotient} \\
\hline
x^2 - 3 & x^4 + 4x^3 - 5x^2 - 12x + 6 \\
 & \underline{x^4 \qquad\quad - 3x^2} \\
 & \qquad 4x^3 - 2x^2 - 12x + 6 \\
 & \qquad \underline{4x^3 \qquad\quad - 12x} \\
 & \qquad\qquad - 2x^2 + 6 \\
 & \qquad\qquad \underline{-2x^2 + 6} \\
 & \qquad\qquad\qquad 0 \longleftarrow \text{remainder}
\end{array}
$$

and as for the verification:

$$
\begin{aligned}
(x^2 - 3)(x^2 + 4x - 2) + 0 &= x^4 + 4x^3 - 2x^2 - 3x^2 - 12x + 6 \\
&= x^4 + 4x^3 - 5x^2 - 12x + 6,
\end{aligned}
$$

there you go!

Exercises

Use long division to find the quotient and the remainder for each case.

1. $\dfrac{x^2+5x+17}{x-3}$

2. $\dfrac{4x^3+3x-5}{x^2+1}$

3. $\dfrac{1-2x^2+3x^3}{x^2-2}$

4. $\dfrac{4x^4+2x^2+1}{2x^2+1}$

5. $\dfrac{5+x^2-4x^3}{x^2+x}$

6. $\dfrac{x^4+x^2+1}{x^2+1}$

7. $\dfrac{7-x^5}{x^2+x+1}$

8. $\dfrac{3x^4-x^3+x}{x-2}$

Answers

1. $\dfrac{x^2+5x+17}{x-3} = (x+8) + \dfrac{41}{x-3}$

2. $\dfrac{4x^3+3x-5}{x^2+1} = 4x + \dfrac{-x-5}{x^2+1}$

3. $\dfrac{1-2x^2+3x^3}{x^2-2} = 3x-2+\dfrac{6x-3}{x^2-2}$

4. $\dfrac{4x^4+2x^2+1}{2x^2+1} = 2x^2 + \dfrac{1}{2x^2+1}$

5. $\dfrac{5+x^2-4x^3}{x^2+x} = -4x+5+\dfrac{5-5x}{x^2+x}$

6. $\dfrac{x^4+x^2+1}{x^2+1} = x^2 + \dfrac{1}{x^2+1}$

7. $\dfrac{7-x^5}{x^2+x+1} = -x^3+x^2-1+\dfrac{x+8}{x^2+x+1}$

8. $\dfrac{3x^4-x^3+x}{x-2} = 3x^3+5x^2+10x+21+\dfrac{42}{x-2}$