SUPPLEMENTARY NOTES

Optimization Problems Involving Cost

RECTANGULAR FIELDS

Example 1: A rancher plans to enclose a rectangular field, next to a road, with fencing along all four sides. The fencing along the road costs \$16 per meter, the fencing along the sides costs \$7 per meter and fencing along the back costs \$5 per meter. The total amount of money available for fencing material is \$10584. Find the dimensions of the field of greatest area.

Solution: Let L be the length (along the road), let W be the width, let A be the area.
Then the total cost is $10584 = 16L + 2(7W) + 5L \;\rightarrow\; 21L + 14W = 10584 \;\rightarrow\; L = 504 - \frac{2}{3}W$

Find the area as a function of W, and then find the derivative.

$$A = LW = \left(504 - \tfrac{2}{3}W\right)W \;\rightarrow\; A(W) = 504W - \tfrac{2}{3}W^2 \;\; (W > 0) \;\rightarrow\; A'(W) = 504 - \tfrac{4}{3}W$$

Set the derivative to zero to find the critical value of W.

$$A'(W) = 0 \;\rightarrow\; 504 - \tfrac{4}{3}W = 0 \;\rightarrow\; W = 378 \;\; (L = 252)$$

Test the increasing/decreasing behaviour of A (i.e. apply the First Derivative Test).

$A'(10) > 0$ and $A'(900) < 0$ $\rightarrow$ The (absolute) maximum point is $(378, A(378))$.

The field of largest possible area is 252 meters (along the road) by 378 meters.

Example 2: A farmer intends to fence off a rectangular field and subdivide it into two equal parts with interior fencing parallel to one side. The fencing along the outside of the field costs \$13/meter. The interior fencing costs \$4/meter. If the total area of the enclosure is to be 7020 square meters, find the dimensions of the field of minimum cost.

Solution: Let L be the length, Let W be the width, let C be the cost. Then the total area is

$$LW = 7020 \;\rightarrow\; L = \frac{7020}{W} = 7020\,W^{-1} \;\; (W > 0)$$

Find the cost as a function of W, and then find the derivative.

$$C = 2(13L) + 2(13W) + 4W = 26\left(7020W^{-1}\right) + 30W \;\rightarrow\; C(W) = 182{,}520W^{-1} + 30W \;\; (W > 0)$$

$$C'(W) = -182{,}520W^{-2} + 30 \;\rightarrow\; C'(W) = \frac{30W^2 - 182{,}520}{W^2} \;\rightarrow\; C'(W) = \frac{30\left(W^2 - 6{,}084\right)}{W^2} \;\; (W > 0)$$

Set the derivative to zero to find the critical value of W.

$$C'(W) = 0 \;\rightarrow\; \frac{30\left(W^2 - 6{,}084\right)}{W^2} = 0 \;\rightarrow\; W = +78 \quad and \quad L = \frac{7020}{W} = \frac{7020}{78} = 90$$

Test the increasing/decreasing behavior of C (i.e. apply the First Derivative Test).

$C'(1) < 0 \;\; and \;\; C'(100) > 0$ $\rightarrow$ The (absolute) minimum point is $(78\;,\; C(78))$.

The field of least cost is 90 meters by 78 meters.

Exercises:

1. A homeowner wants to fence off a rectangular garden plot next to the street. The fencing along the street costs \$14 per meter. The fencing along the other three sides costs \$10 per meter. The total amount of money available for fencing material is \$240. Find the dimensions of the garden of maximum area.

2. A rancher plans to enclose a rectangular field next to a road (there will be no fence along the road). The cost of fencing material along the sides (perpendicular to the road) is \$6/meter. The cost of fencing material along the back is \$8/meter. If the total area of the field is to be 864 square meters, find the dimensions of the field of minimum cost.

3. A company plans to enclose a rectangular area next to a warehouse and divide it into three equal parts. The fencing along the outside costs \$12/meter and the interior fencing costs \$9/meter. If the total amount of money available for fencing material is \$336 find the dimensions of the field of greatest area.

4. A farmer intends to fence off a rectangular field next to a highway and divide it into four equal parts using interior fencing parallel to the highway. Fencing along the highway costs \$15 per meter. Fencing along the other three sides, and the interior fencing, costs \$11 per meter. The total area of the field is to be 5192 square meters. Find the dimensions of the field of least cost.

5. A rancher intends to enclose a rectangular field next to a canal (no fencing along the canal). He will divide the field into three equal parts with fencing perpendicular to the canal. The fencing along the outside costs \$28/meter and the interior fencing costs \$21/meter. If a total of \$8232 is available for fencing material, find the dimensions of the largest possible field.

Answers to Exercises:

1. 5 m (along the street) by 6 m

2. 36 m (along the building) by 24 m.

3. 7 m by 4 m (along interior fencing)

4. 44 m (along the highway) by 118 m

5. 147 m (along the canal) by 42 m

RECTANGULAR BOXES

Example 1: A closed rectangular box is to be constructed of material that costs 80¢/sq.cm for the top, 40¢/sq.cm for the sides and 30¢/sq.cm for the bottom. If box will have a square base and the total volume will be 5632 cubic centimeters, find the dimensions of the box of least cost.

Solution: Let x be the length of each side (square base), h the height, and C the (total) cost. Then the total volume is $5632 = x \cdot x \cdot h \;\rightarrow\; x^2 h = 5632 \;\rightarrow\; h = 5632x^{-2}\ (x>0)$

Find the cost as a function of x, and then find the derivative.

$$C = 80x^2 + 4(40xh) + 30x^2 = 110x^2 + 160x(5632x^{-2}) \rightarrow C(x) = 110x^2 + 901120x^{-1}\ (x>0)$$

$$C'(x) = 220x - 901120x^{-2} \;\rightarrow\; C'(x) = \frac{220x^3 - 901120}{x^2} \;\rightarrow\; C'(x) = \frac{220(x^3 - 4096)}{x^2}\ (x>0)$$

Set the derivative to zero to find the critical value of x.

$$C'(x) = 0 \;\rightarrow\; \frac{220(x^3 - 4096)}{x^2} = 0 \;\rightarrow\; x = 16\ (h = 22)$$

Test the increasing/decreasing behaviour of C (i.e. apply the First Derivative Test).

$$C'(1) < 0 \text{ and } C'(50) > 0 \;\rightarrow\; \text{The (absolute) minimum point is } (16, C(16)).$$

The box of least cost is 16 cm by 16 cm by 22 cm (high).

Example 2: An open rectangular container is to have a base whose length is three times the width. The cost of material is \$4 per ft^2 for the sides and \$6 per ft^2 for the bottom. If \$3456 will be spent to make this box, find the dimensions of the box of maximum volume.

Solution: Let x be the width ($3x$ is the length), h the height, and V the volume. Then the total cost is $3456 = 2(4 \cdot x \cdot h) + 2(4 \cdot 3x \cdot h) + 6(3x \cdot x) = 32xh + 18x^2$

$$\rightarrow\; 32xh = 3456 - 18x^2 \;\rightarrow\; h = 108x^{-1} - \tfrac{9}{16}x\ (x>0)$$

Find the volume as a function of x, and then find the derivative.

$$V = 3x \cdot x \cdot h = 3x^2\left(108x^{-1} - \tfrac{9}{16}x\right) \;\rightarrow\; V(x) = 324x - \tfrac{27}{16}x^3\ (x>0)$$

$$V'(x) = 324 - \tfrac{81}{16}x^2 \;\rightarrow\; V'(x) = \tfrac{81}{16}(64 - x^2) \;\rightarrow\; V'(x) = \tfrac{81}{16}(8-x)(8+x)\ (x>0)$$

Set the derivative to zero to find the critical value of x.

$$V'(x) = 0 \;\rightarrow\; \tfrac{81}{16}(8-x)(8+x) = 0 \;\rightarrow\; x = 8\ (h = 9)$$

Test the increasing/decreasing behaviour of V (i.e. apply the First Derivative Test).

$$V'(1) > 0 \text{ and } V'(10) < 0 \;\rightarrow\; \text{The (absolute) maximum point is } (8, V(8)).$$

The container of maximum volume is 8 feet by 24 feet by 9 feet (high).

Exercises:

1. An large open rectangular container is to be constructed of material that costs \$36/sq.m for the sides and \$45/sq.m for the bottom. The length of the base will be 2 times the width, and the total volume will be 5.625 cubic meters. Find the dimensions of the box of least cost.

2. A closed rectangular box is to have a square base. The cost of material is \$0.03 per sq.cm. for the sides and \$0.02 per sq.cm. for the top and bottom. If \$27.00 will be spent to make this box, find the dimensions of the box of maximum volume.

3. A closed rectangular box is to be constructed of material that costs \$4 per square feet for the top and \$3 per square feet for the sides and bottom. The length of the base will be 3 times the width, and the total volume will be 10.5 cubic feet. Find the dimensions of the box of least cost.

4. An open rectangular box is to be constructed of material that costs \$0.10/sq.in. for three sides and \$0.06/sq.in. for the back (one of the larger sides) and the bottom. The length of the base is 5 times the width and \$32.40 will be spent to make this box. Find the dimensions of the box of maximum volume.

Answers to Exercises:

1. 1.5 m by 3 m by 1.25 m high

2. 15 cm by 15 cm by 10 cm high

3. 1.26 ft. by 3.78 ft. by 2.2 ft high

4. 6 in. by 30 in. by 3.6 in. high