Dawson College,
Mathematics Department,
Fall Semester 2024,
Dec. 12-th, 9:30-12:30.

Final Exam 201-MA2-DW Calculus 2,
Sections 1, 2, 3.

Instructors: A. Jimenez and S. Soltuz.

Student Name: SOLUTIONS

Student I.D. #: ____________________

Instructor Name: ____________________

INSTRUCTIONS:

- Print your name and student number in the space provided above.
- Attempt all questions. Show all your work clearly and justify your answers.
- All questions are to be answered directly on the examination paper in the space provided. If you need more space for your answer use the back of the page.
- You are only permitted to use the Sharp EL-531** calculator.
- Verify that your final examination copy has a total of 12 questions on 13 pages, including this cover page.
- Please ensure that you have a complete exam package before starting.
- The examination must be returned intact.

Question 1 (9=4+5 Marks) A bank loan of 1200 \$ is repaid in annual instalments of 100 \$ over 12 years, plus 10% interest paid on the unpaid balance at the start of each year.

a) How much interest is to be paid in the 4-th year?

b) What is the total amount of interest paid?

$$a_1 = 1200 \cdot \frac{10}{100} = 120$$

$$a_2 = 1100 \cdot \frac{10}{100} = 110$$

$$a_3 = 1000 \cdot \frac{10}{100} = 100$$

$$a_4 = 900 \cdot \frac{10}{100} = 90$$

$$\ldots$$

$$a_{12} = 10$$

$$S = 10 + 20 + \ldots + 110 + 120$$

$$S = \frac{n(a_1 + a_{12})}{2} = \frac{12 \cdot (10 + 120)}{2} = \frac{12 \cdot 130}{2} =$$

$$= \underline{780}$$

Question 2 (8 Marks) Compute

$$\int_e^{+\infty} \frac{1}{x\ (\ln x)^2} dx. \qquad = \int_1^{\infty} \frac{1}{u^2}\, du =$$

$$u = \ln x \qquad\qquad u_1 = \ln e = 1$$

$$du = (\ln x)' dx \qquad\qquad u_2 = \ln \infty = \infty$$

$$du = \frac{1}{x} dx$$

$$= \left.\frac{u^{-2+1}}{-2+1}\right|_1^{\infty} = -\left.\frac{1}{u}\right|_1^{\infty} = -\underbrace{\frac{1}{\infty}}_{=0} + \frac{1}{1} = 1$$

Question 3 (8=4+4 Marks) For the function

$$f(x) = x^3 + 2x - 3,$$

compute

a) the value of this integral over $[1, 3]$

b) the average value over $[1, 3]$.

a) $$\int_1^3 (x^3+2x-3)\,dx = \left(\frac{x^4}{4} + x^2 - 3x\right)\Bigg|_1^3 =$$

$$= \left(\frac{3^4}{4} + \cancel{3^2} - \cancel{3\cdot 3}\right) - \left(\frac{1}{4} + \underbrace{\cancel{1} - \cancel{3}}_{-2}\right) = \frac{3^4}{4} - \left(\frac{1}{4} - \frac{8}{4}\right)$$

$$= \frac{3^4}{4} + \frac{7}{4} = \frac{81+7}{4} = \frac{88}{4} = 22$$

b) $$\text{Average} = \frac{1}{3-1} \int_1^3 (x^3+2x-3)\,dx =$$

$$= \frac{1}{2} \cdot 22 = \underline{\underline{11}}$$

Question 4 (8 Marks) Compute

$$\int x\sqrt{x-4}dx$$

$$\int x\sqrt{x-4}\,dx = \int (u+4)\sqrt{u}\,du =$$

$u = x-4$
$du = dx$
$x = u+4$

$$= \int \left(u^{\frac{3}{2}} + 4u^{\frac{1}{2}}\right)du =$$

$$= \frac{u^{\frac{3}{2}+1}}{\frac{3}{2}+1} + 4\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C$$

$$= \frac{u^2\sqrt{u}}{\frac{5}{2}} + 4\frac{u\sqrt{u}}{\frac{3}{2}} + C =$$

$$= \frac{2}{5}(x-4)^2\sqrt{x-4} + \frac{8}{3}(x-4)\sqrt{x-4} + C$$

OR $\frac{2}{5}(x-4)^{\frac{5}{2}} + \frac{8}{3}(x-4)^{\frac{3}{2}} + C$

Question 5 (9=4+5 Marks) The daily marginal revenue function associated with producing and selling clock is given by

$$R'(x) = -0.004x + 15,$$

a) Determine the Revenue function $R(x)$ associated with the above marginal revenue function, if $R(0) = 0$.

b) Determine the Demand function that relates the whole sale unit price with quantity of clocks.

a) $R(x) = \int R' dx = \int (-0.004x + 15) dx =$

$= -0.004 \frac{x^2}{2} + 15x + C =$

$= -0.002x^2 + 15x + C$

$R(0) = 0 \Rightarrow C = 0$, $\underline{\underline{R(x) = -0.002x^2 + 15x}}$

b) $R = p \cdot x = D(x) \cdot x$

but $R = (-0.002x + 15)x$ $\Big\} \Rightarrow$

$\Rightarrow D(x) = -0.002x + 15$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x$$

Question 6 (9 Marks) Use the Riemann sums to compute the following defined integral

$$\int_0^1 (2x+5)\,dx.$$

$$\Delta x = \frac{b-a}{n}, \quad x_k = a + k\Delta x$$

$$\Delta x = \frac{1}{n}$$

$$x_k = 0 + \frac{k}{n} = \frac{k}{n}$$

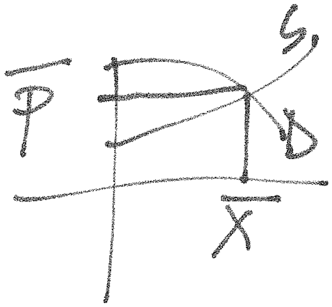
$$\int_0^1 (2x+5)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{2k}{n} + 5\right)\frac{1}{n} =$$

$$= \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{2}{n^2}k + \sum_{k=1}^{n} \frac{5}{n} \right) =$$

$$= \lim_{n\to\infty} \left(\frac{2}{n^2} \sum_{k=1}^{n} k + \frac{5}{n} \sum_{k=1}^{n} 1 \right) =$$

$$= \lim_{n\to\infty} \left(\frac{2}{n^2} \cdot \frac{n(n+1)}{2} + \frac{5}{n} \cdot n \right) = 1 + 5 = 6$$

$$\int_0^1 (2x+5)\,dx = x^2\Big|_0^1 + 5x\Big|_0^1 = 1^2 + 5 = 6$$



Question 7. (11=5+6 Marks) If the Demand function is given by

$$p = -0.1x^2 + 60,$$

and the Supply function by

$$p = 0.1x^2 + 2x + 20,$$

where p is the price/unit, and x is number of units in thousand.

a) Determine the equilibrium

b) Determine Consumers' Surplus.

$400+3200$
$=$

$$-0.1x^2+60 = 0.1x^2+2x+20$$
$$0 = 0.2x^2+2x-60+20$$
$$0 = 0.2x^2+2x-40 \;|\cdot 10$$
$$0 = 2x^2+20x-400$$

$$\Delta = 20^2+4\cdot 2\cdot 400$$
$$\Delta = 3600 = 60^2$$

$$x_1 = \cancel{\frac{-20-60}{2\cdot 2}} \qquad x_2 = \frac{-20+60}{2\cdot 2} = \frac{40}{4} = \underline{\underline{10}}$$

$$\underline{\bar{x}=10} \qquad \bar{p} = -0.1\cdot 100+60 = -10+60 = \underline{\underline{50}}$$

$$CS = \int_0^{10}(-0.1x^2+60)\,dx - 50\cdot 10 = -0.1\cdot\frac{x^3}{3} + 60x\Big|_0^{10} - 500$$

$$= -0.1\,\frac{1000}{3} + 600 - 500 = -\frac{100}{3} + 600 - 500 = \underline{\underline{66.67}}$$

$$\frac{200}{3} \qquad \left(\frac{200}{3}\right)$$

Question 8 (9 Marks) Establish the indetermination and compute

$$\lim_{x \to 0+} x^2 \ln x.$$

$0 \cdot \ln 0+ = 0 \cdot (-\infty)$ / $\frac{-\infty}{+\infty}$

$$\lim_{x \to 0+} x^2 \ln x = \lim_{x \to 0+} \frac{\ln x}{\frac{1}{x^2}} \overset{L'H}{=} \lim_{x \to 0+} \frac{(\ln x)'}{\left(\frac{1}{x^2}\right)'}$$

$$= \lim_{x \to 0+} \frac{\frac{1}{x}}{\frac{-2}{x^3}} = \lim_{x \to 0+} \left(-\frac{1}{2}\right) \cdot \frac{x^3}{x} =$$

$$= \lim_{x \to 0+} \left(-\frac{1}{2}\right) x^2 = 0$$

Questioin 9 (8 Marks) Compute

$$\int (2x+1)\ e^{3x} dx.$$

$$\int e^{3x} \cdot (2x+1) dx = \frac{e^{3x}}{3}(2x+1) - \int \frac{e^{3x}}{3} \cdot (2x+1)' dx$$

$$= \frac{e^{3x}}{3}(2x+1) - \frac{2}{3}\int e^{3x} dx =$$

$$= \frac{e^{3x}}{3}(2x+1) - \frac{2}{3} \cdot \frac{e^{3x}}{3} + C =$$

$$= \frac{e^{3x}}{3}(2x+1) - \frac{2}{9} e^{3x} + C$$

$$y' = \frac{dy}{dx}$$

Question 10 (9 Marks) Solve the initial value problem

$$y' = \frac{3x^2}{y},$$
$$y(1) = 2.$$

$$\frac{dy}{dx} = \frac{3x^2}{y}$$

$$y\,dy = 3x^2\,dx$$

$$\int y\,dy = 3\int x^2\,dx$$

$$\frac{y^2}{2} = x^3 + C$$

$$\frac{2^2}{2} = 1 + C$$

$$2 = 1 + C$$

$$\underline{C = 1}$$

$$\frac{y^2}{2} = x^3 + 1$$

Question 11 (8 Marks) Find K such that the function

$$f(x) = K(6 - x)$$

is a probaility density function on $[0, 6]$.

$$\int_0^6 K(6-x)\,dx = 1$$

$$K\left(6x\Big|_0^6 - \frac{x^2}{2}\Big|_0^6\right) = 1$$

$$K\left(6\cdot 6 - \frac{6^2}{2}\right) = 1$$

$$K\left(\frac{2\cdot 6^2 - 6^2}{2}\right) = 1$$

$$K\,\frac{6^2}{2} = 1$$

$$K \cdot 18 = 1 \qquad \underline{\underline{K = \frac{1}{18}}}$$

Question 12 (4 Marks)Let Z be the standard normal variable. Find the probability if

$$P\left(0 < z < 0.82\right).$$

$P(z < 0.82) = 0.7932$

$$\begin{array}{r} 0.7932 \\ -\ 0.5 \\ \hline 0.2932 \end{array}$$

$P(0 < z < 0.82) = 0.2932$