DAWSON COLLEGE
Mathematics Department
Final Examination
Calculus I: Social Sciences
201-MA1-DW Sections 01, 02, 03, 04, 05, 06, 07, 08, 09, 10
Dec 12th, 2024
14:00-17:00

Student Name ____

Student I.D. # ____

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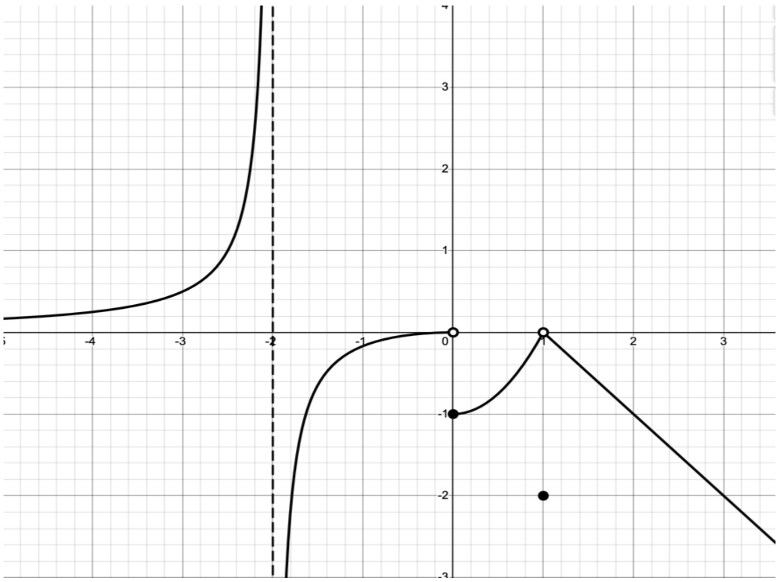
Instructions:
- Print your name and student ID number in the space provided above.
- All questions are to be answered directly on the examination paper in the provided space. If you run out of room for an answer, continue on the back of the page and please indicate that you have done so.
- Only the Sharp EL-531X, XG or XT calculators are permitted.
- This examination consists of 15 questions.
- This exam booklet must be returned intact.
- Good Luck!

Question #	Mark
1	/7
2	/10
3	/6
4	/6
5	/9
6	/5
7	/4
8	/5
9	/6
10	/5
11	/6
12	/6
13	/7
14	/6
15	/12
Total	/100

Q1*: [7 marks]* Given the graph of a function f, answer the following questions. If the limit of the function does not exist, write $-\infty$ or ∞, where possible.



a. $\lim_{x\to-\infty} f(x) = 0$

b. $\lim_{x\to-2^+} f(x) = -\infty$

c. $\lim_{x\to 0^-} f(x) = 0$

d. $\lim_{x\to 1} f(x) = 0$

e. $\lim_{x\to 2} f(x) = -1$

f. Write the equation of the vertical asymptote. $x = -2$

g. Is the function continuous at $x = 1$? Justify your answer based on the three conditions of continuity. No, $\lim_{x\to 1} f(x) = 0 \neq f(1) = -2$

Q2. *[5+5 marks]*. Evaluate the following limits:

a.

$$\lim_{x\to 5}\frac{\sqrt{x+4}-3}{(x-5)} = \lim_{x\to 5}\frac{\sqrt{x+4}-3}{(x-5)}\cdot\frac{\sqrt{x+4}+3}{\sqrt{x+4}+3} = \lim_{x\to 5}\frac{x+4-9}{(x-5)}\cdot\frac{1}{\sqrt{x+4}+3}$$

$$= \lim_{x\to 5}\frac{\cancel{x-5}}{\cancel{(x-5)}}\cdot\frac{1}{\sqrt{x+4}-3} = \frac{1}{\sqrt{9}+3} = \frac{1}{6}$$

b.

$$\lim_{x\to -1}\frac{x^2+x}{3x^2+7x+4} = \lim_{x\to -1}\frac{x\cancel{(x+1)}}{\cancel{(x+1)}(3x+4)} = \frac{-1}{1} = -1$$

Q3. *[5+1 marks]* The function that gives the profit of a certain product is given by

$$P(x) = \begin{cases} 5.25 & if\ \ 0 < x \le 1000 \\ 5250 + k(x-1000) & if\ \ 1000 < x \le 2500 \\ 5325 + 0.15(x-2500) & if\ \ 2500 < x \le 4000 \end{cases}$$

where x is the quantity sold and $P(x)$ is measured in dollars.

a. Find the value of k which makes the function continuous at $x = 2500$.

$$\lim_{x\to 2500^-} P(x) = 5250 + k(2500-1000) = 5250 + 1500k = P(2500)$$

$$\lim_{x\to 2500^+} P(x) = 5325 + 0.15(2500-2500) = 5325$$

For continuity at $x = 2500$:

$$5250 + 1500k = 5325;\ 1500k = 75;\ \ k = 0.05$$

b. What is the profit earned when 3500 units are sold?

$$P(3500) = 5325 + 0.15(3500-2500) = 5325 + 0.15(1000) = \$5475$$

Q4. *[6 marks]* Use the limit definition of the derivative to find $f'(x)$.

$$f(x) = 3 - 5x^2$$

$$f'(x) = \lim_{h\to 0}\frac{3-5(x+h)^2-(3-5x^2)}{h} = \lim_{h\to 0}\frac{3-5(x^2+2xh+h^2)-(3-5x^2)}{h}$$

$$= \lim_{h\to 0}\frac{-5h(2x+h)}{h} = \lim_{h\to 0} -5(2x+h) = -10x$$

Q5. *[1+5+3 marks]* A newly released smartphone operating system gives users an immediate notice to update but no further reminders. The percent P of users that have installed the new updates after t days is given by:

$$P(t) = \frac{99\,t^2}{t^2 + 50}$$

a. What percentage of users have the new updates installed after 10 days?

$$P(10) = \frac{99\,(10)^2}{(10)^2 + 50} = \frac{9900}{150} = 66 = 66\%$$

b. Determine the rate of change, $P'(t)$, and evaluate $P'(10)$.

$$P'(t) = \frac{198t(t^2 + 50) - 99\,t^2(2t)}{(t^2 + 50)^2} = \frac{\cancel{198t^2} + 9900t - \cancel{198\,t^2}}{(t^2 + 50)^2} = \frac{9900t}{(t^2 + 50)^2}$$

$$P'(10) = \frac{9900(10)}{((10)^2 + 50)^2} = \frac{99000}{150^2} = 4.4 = 4.4\%$$

c. What happens to $P(t)$ as $t \to \infty$. Find the answer and interpret the result.

$$\lim_{t\to\infty} \frac{99\,t^2}{t^2 + 50} = 99$$

99% will have installed updates.

Q6. *[3+2 marks]* The demand x of a certain commodity is related to the unit price p (in dollars) by the equation

$$p = 500\left(1 - \frac{2}{2 + e^{-0.0001x}}\right) = 500 - \frac{1000}{2 + e^{-0.0001x}} = 500 - 1000(2 + e^{-0.0001x})^{-1}$$

a. Find the rate of change in price when $x = 100$.

$$p'(x) = 1000(2 + e^{-0.0001x})^{-2}(e^{-0.0001x})(-.0001) = \frac{-0.1e^{-0.0001x}}{(2 + e^{-0.0001x})^2}$$

$$p'(100) = \frac{-0.1e^{-0.01}}{(2 + e^{-0.01})^2} = -0.011$$

b. Determine the revenue when $x = 100$

$$R(x) = 500x\left(1 - \frac{2}{2 + e^{-0.0001x}}\right);\;\; R(100) = 50\,000\left(1 - \frac{2}{2 + e^{-0.01}}\right) = 16\,555.74$$

$$= \$16\,555.74$$

Q7. *[4 marks]* Evaluate the derivative of the following function, using logarithmic differentiation.

$$f(x) = (\tan 3x)^{4x}$$

$\ln f(x) = \ln(\tan 3x)^{4x} = 4x \ln(\tan 3x)$

$$\frac{f'(x)}{f(x)} = 4\ln(\tan 3x) + \frac{4x \cdot 3(\sec^2 3x)}{\tan 3x}$$

$$f'(x) = (\tan 3x)^{4x} \left[4 \ln(\tan 3x) + \frac{12x \sec^2(3x)}{\tan 3x}\right]$$

Q8. *[5 marks]* For the function $f(x) = \sin(x) \ln(x)$, find $f''(x)$.

$$f'(x) = \cos(x) \ln(x) + \frac{\sin(x)}{x}$$

$$f''(x) = -\sin(x) \ln(x) + \frac{\cos x}{x} + \frac{x \cos(x) - \sin(x)}{x^2} = -\sin(x) \ln(x) + \frac{2 \cos(x)}{x} - \frac{\sin(x)}{x^2}$$

Q9. *[2+2+2 marks]* The cost $C(x)$, in dollars, of a certain product is related to the number of units x by the equation

$$C(x) = 23500 + 850x - 0.02x^2$$

a. Determine the marginal cost of producing the 100th unit.

$$C'(x) = 850 - 0.04x; \;\; C'(99) = 850 - 0.04(99) = 846.04 = \$846.04$$

b. Find the actual cost of producing the 100th unit.

$$C(100) - C(99) = 850(100 - 99) - 0.02(100^2 - 99^2) = 846.02 = \$846.02$$

c. Find the average cost function $\overline{C}(x)$.

$$\overline{C}(x) = \frac{23500}{x} + 850 - 0.02x$$

Q10. *[2+3 marks]* The population of Canadians aged 55 and older as a percentage of the total population is approximated by the function

$$f(t) = 15\sqrt{0.7t + 6.2} \qquad 0 \le t \le 30$$

where t is measured in years, with $t = 0$ corresponding to the beginning of year 2020.

a. What was the percentage of the population aged 55 or older at the beginning of 2024?

$$f(4) = 15\sqrt{0.7(4) + 6.2} = 45 = 45\%$$

b. At what rate did the percentage of the population of aged 55 or older increase at the beginning of 2024?

$$f'(t) = \frac{5.25}{\sqrt{0.7t + 6.2}}; \;\; f'(4) = \frac{5.25}{\sqrt{0.7(4) + 6.2}} = 1.75$$

.

It increased by a rate of 1.75%

Q11. *[6 marks]* The demand equation for a certain brand of tablets is given by

$$150x^2 + 4p^2 = 3696$$

where x represents the quantity (in thousands) of tablets demanded each week when the unit price is p dollars. How fast is the quantity demanded increasing when the unit price is $18, and the price drops at the rate of $0.25 per tablet per week?

$$150x^2 + 4p^2 = 3696$$

$$300x\frac{dx}{dt} + 8p\frac{dp}{dt} = 0; \;\; 300x\frac{dx}{dt} + 8(18)(-0.25) = 0; \;\; x\frac{dx}{dt} = \frac{36}{300} = \frac{6}{50}; \;\frac{dx}{dt} = \frac{6}{50x}$$

Need to find x when $p = 18$

$$150x^2 + 4(18)^2 = 3696; \;\; 150x^2 + 1296 = 3696; \;\; x^2 = \frac{2400}{150} = 16; x = 4$$

$$\frac{dx}{dt} = \frac{6}{50(4)} = 0.03$$

Measured in thousands so that there is an increase of 30 units.

Q12. *[4+2 marks]* Consider the function $3xy^2 + 2y = x + 2$

a. Use implicit differentiation to find y'.

$3y^2 + 3x(2yy') + 2y' = 1$

$3y^2 + 6xyy' + 2y' = 1$

$6xyy' + 2y' = 1 - 3y^2$

$$y' = \frac{1 - 3y^2}{6xy + 2}$$

b. Use your solution from part a) to find the equation of the tangent line to the curve at the point $P(0,1)$.

$$y' = \frac{1 - 3(1)^2}{6(0)(1) + 2} = \frac{-2}{2} = -1$$

Tangent line: $y = -1(x - 1);\ y = -x + 1$

Q13. *[7 marks]* The quantity demanded each month of a robotic part is given by

$$p = -0.00035x + 8 \qquad 0 \le x \le 15\,000$$

where p denotes the unit price in dollars and x is the number of units demanded. The total monthly cost (in dollars) of producing x units is given by

$$C(x) = 500 + 3x - 0.00015x^2 \qquad 0 \le x \le 16\,000$$

How many units should be sold each month to maximize profits, where $P(x) = R(x) - C(x)$? What is the maximum profit? Justify that the value found is the maximum.

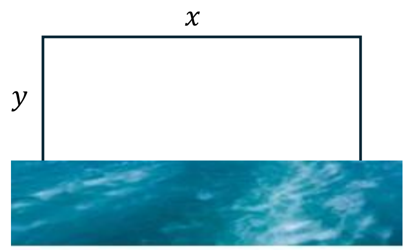
$P(x) = -0.00035x^2 + 8x - (500 + 3x - 0.00015x^2) = -0.0002x^2 + 5x - 500$

$P'(x) = -0.0004x + 5 = 0$ when $x = 12\,500$

To justify this is the maximum: $x \in [0, 15\,000]$

$\mathrm{P}(0) = -500$; $\mathrm{P}(15000) = 29500$; $\mathrm{P}(12500) = 30\,750$. Maximum profit $30 750 for 12 500 units

Q14. *[6 marks]* A farmer wants to fence off a rectangular field adjacent to a river. The fencing along the riverside does not require a fence, while the other three sides must be fenced. The cost of fencing along the length x of the field is \$20 per meter, and the cost along the width y is \$12 per meter. The total amount of money available for fencing is \$360. Find the dimensions of the field that maximizes the area.



Constraint: $20x + 24y = 360$:

$$y = \frac{360 - 20x}{24} = 15 - \frac{5}{6}x$$

Objective Function: $A = xy = x\left(15 - \frac{5}{6}x\right)$; $0 \leq x \leq 18$

$A(x) = 15x - \frac{5}{6}x^2$; $A'(x) = 15 - \frac{5}{3}x = 0$, When $x = 9$.

To check this is the maximum:

$A(0) = 0$; $A(18) = 0$; $A(9) = 67.5$. Maximum area when $x = 9\ m$, $y = 7.5\ m$

Q15. *[3+4+3+2 marks]* Let

$$f(x) = \frac{1}{x(x-1)}$$

Where the first and second derivatives are:

$$f'(x) = \frac{1-2x}{x^2(x-1)^2}$$

and

$$f''(x) = \frac{2(3x^2-3x+1)}{x^3(x-1)^3}$$

a. Find the equation(s) of the horizontal and vertical asymptote(s) of $f(x)$, if any.
Vertical asymptotes $x = 0$ and $x = 1$:

$$\lim_{x\to 0^-} \frac{1}{x(x-1)} = \infty; \quad \lim_{x\to 0^+} \frac{1}{x(x-1)} = -\infty; \quad \lim_{x\to 1^-} \frac{1}{x(x-1)} = -\infty; \quad \lim_{x\to 1^+} \frac{1}{x(x-1)} = \infty$$

Horizontal asymptote $y = 0$:

$$\lim_{x\to -\infty} \frac{1}{x(x-1)} = 0; \quad \lim_{x\to \infty} \frac{1}{x(x-1)} = 0$$

b. Determine the intervals where the function $f(x)$ is increasing/decreasing, and its relative extrema, if any.
$f(x)$ increasing on $(-\infty, 0) \cup \left(0, \frac{1}{2}\right)$, decreasing on $\left(\frac{1}{2}, 1\right) \cup (1, \infty)$. Relative max.: $\left(\frac{1}{2}, -4\right)$

c. Determine the intervals where the graph of the function $f(x)$ is concave up/concave down, and its inflection point(s), if any.
The graph of $f(x)$ is concave up on $(-\infty, 0) \cup (1, \infty)$, concave down on $(0,1)$. No inflection points

d. Sketch the graph of $f(x)$.