L'Hôpital's Rule

Let a represent a (finite) real number or ∞ or $-\infty$.

Suppose $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$. Then $\lim_{x\to a} \frac{f(x)}{g(x)}$ may or may not exist, and we call the limit an **indeterminate form of type** $\frac{0}{0}$.

Similarly, suppose $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$. Then $\lim_{x\to a} \frac{f(x)}{g(x)}$ may or may not exist, and we call the limit an **indeterminate form of type** $\frac{\infty}{\infty}$.

L'Hôpital's Rule

Suppose f and g are differentiable functions and that $\lim_{x\to a} \frac{f(x)}{g(x)}$ is an indeterminate form of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$ provided that the limit on the right exists.

Note:

- If the limit on the right does not exist (is ∞ or $-\infty$) then the limit on the left does not exist.
- If the limit on the right still gives an indeterminate form of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$, we keep using L'Hôpital's Rule (or a factoring or rationalizing technique) as needed.
- There exist other types of indeterminate forms, such as $0 \cdot \infty$ and $\infty - \infty$, to be discussed later in this section.

Example 1 Find $\lim_{x\to 2} \frac{5x^3-13x^2+6x}{4x^2-13x+10}$

Since $\lim_{x\to 2}(5x^3 - 13x^2 + 6x) = 0$ and $\lim_{x\to 2}(4x^2 - 13x + 10) = 0$ we can apply l'Hôpital's Rule.

$$\lim_{x\to 2} \frac{5x^3 - 13x^2 + 6x}{4x^2 - 13x + 10} = \lim_{x\to 2} \frac{15x^2 - 26x + 6}{8x - 13} = \frac{15(2)^2 - 26(2) + 6}{8(2) - 13} = \frac{14}{3}$$

Example 2 Find $\lim_{x\to\infty} \frac{10x+5}{3x^2-7x+4}$

Since $\lim_{x\to\infty}(10x + 5) = \infty$ and $\lim_{x\to 2}(3x^2 - 7x + 4) = \infty$ we can apply l'Hôpital's Rule.

$$\lim_{x\to\infty} \frac{10x + 5}{3x^2 - 7x + 4} = \lim_{x\to\infty} \frac{10}{6x - 7} = 0$$

Example 3 Find $\lim_{x\to 0} \frac{e^x}{1-\cos x}$

Since $\lim_{x\to 0}(e^x) = 1$ and $\lim_{x\to 0}(1 - \cos x) = 0$ we can NOT apply l'Hôpital's Rule.

$$\lim_{x\to 0} \frac{e^x}{1 - \cos x} = \infty$$

Example 4 Find $\lim_{x\to 0} \frac{\cos x - 1}{e^x - 1}$

Since $\lim_{x\to 0}(\cos x - 1) = 0$ and $\lim_{x\to 0}(e^x - 1) = 0$ we can apply l'Hôpital's Rule.

$$\lim_{x\to 0} \frac{\cos x - 1}{e^x - 1} = \lim_{x\to 0} \frac{-\sin x}{e^x} = \frac{-\sin 0}{e^0} = \frac{0}{1} = 0$$

Example 5 Find $\lim_{x\to 1^+} \frac{7\sqrt{x-1}}{\sin(x-1)}$

Since $\lim_{x\to 1^+}\left(7\sqrt{x - 1}\right) = 0$ and $\lim_{x\to 1^+} \sin(x - 1) = 0$ we can apply l'Hôpital's Rule.

$$\lim_{x\to 1^+} \frac{7\sqrt{x - 1}}{\sin(x - 1)} = \lim_{x\to 1^+} \frac{\frac{7}{2\sqrt{x - 1}}}{\cos(x - 1)} = \lim_{x\to 1^+} \frac{7}{2\sqrt{x - 1}\cos(x - 1)} = \infty$$

Example 6 Find $\lim_{x\to \infty} \frac{3\ln(5x+3)}{2\ln(x+4)}$

Since $\lim_{x\to \infty} 3\ln(5x + 3) = \infty$ and $\lim_{x\to \infty} 2\ln(x + 4) = \infty$ we can apply l'Hôpital's Rule.

$$\lim_{x\to \infty} \frac{3\ln(5x + 3)}{2\ln(x + 4)} = \lim_{x\to \infty} \frac{\frac{3(5)}{5x + 3}}{\frac{2}{x + 4}} = \lim_{x\to \infty} \frac{15(x + 4)}{2(5x + 3)}$$

The limit on the right is also indeterminate (type $\frac{\infty}{\infty}$), so we can apply l'Hôpital's Rule again.

$$\lim_{x\to \infty} \frac{3\ln(5x + 3)}{2\ln(x + 4)} = \lim_{x\to \infty} \frac{15(x + 4)}{2(5x + 3)} = \lim_{x\to \infty} \frac{15}{10} = \frac{3}{2}$$

Other types of indeterminate forms

In the event that the limit $\lim_{x\to a} f(x)g(x)$ produces an **indeterminate form of type** $0 \cdot \infty$, we can convert it into an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by writing the product $f(x)g(x)$ as a quotient $f(x)g(x) = \frac{f(x)}{\frac{1}{g(x)}}$ or $f(x)g(x) = \frac{g(x)}{\frac{1}{f(x)}}$.

Example 7 Find $\lim_{x\to 0^+} x \ln x$

Since $\lim_{x\to 0^+} x = 0$ and $\lim_{x\to 0^+} \ln x = -\infty$ the limit is an indeterminate form of type $0 \cdot \infty$. We must first convert this product into quotient $\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}}$, which gives an indeterminate form of type $\frac{\infty}{\infty}$. Using l'Hôpital's Rule, we have:

$$\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0^+} -x = 0$$

Example 8 Find $\lim_{x\to\infty} x \tan\left(\frac{1}{x}\right)$

Since $\lim_{x\to\infty} x = \infty$ and $\lim_{x\to\infty} \tan\left(\frac{1}{x}\right) = 0$ the limit is an indeterminate form of type $0 \cdot \infty$. We can easily convert it into the $\lim_{x\to\infty} \frac{\tan\left(\frac{1}{x}\right)}{\frac{1}{x}}$ which gives an indeterminate form of type $\frac{0}{0}$. Using l'Hôpital's Rule, we have:

$$\lim_{x\to\infty} x \tan\left(\frac{1}{x}\right) = \lim_{x\to\infty} \frac{\tan\left(\frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{-1}{x^2}\sec^2\left(\frac{1}{x}\right)}{\frac{-1}{x^2}} = \lim_{x\to\infty} \sec^2\left(\frac{1}{x}\right) = 1$$

Example 9 Find $\lim_{x\to\infty} x^3 e^{-x^2}$

It is not difficult to see that the limit is an indeterminate form of type $0 \cdot \infty$. We can easily convert it into the quotient $\lim_{x\to\infty} \frac{x^3}{e^{x^2}}$ which gives an indeterminate form of type $\frac{\infty}{\infty}$. Using l'Hôpital's Rule twice, we obtain:

$$\lim_{x\to\infty} x^3 e^{-x^2} = \lim_{x\to\infty} \frac{x^3}{e^{x^2}} = \lim_{x\to\infty} \frac{3x^2}{2xe^{x^2}} = \lim_{x\to\infty} \frac{3x}{2e^{x^2}} = \lim_{x\to\infty} \frac{3}{4xe^{x^2}} = 0$$

In the event that the limit $\lim_{x \to a}[f(x) - g(x)]$ produces an **indeterminate form of type** $\infty - \infty$, we can convert it into a type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by factoring out a common factor, by rationalization, or by using a common denominator.

Example 10 Find $\lim_{x \to \infty}(x - x^2)$

Since $\lim_{x \to \infty} x = \infty$ and $\lim_{x \to \infty} x^2 = \infty$ the limit is an indeterminate form of type $\infty - \infty$. We convert this by factoring a common factor:

$$\lim_{x \to \infty}(x - x^2) = \lim_{x \to \infty} x(1 - x) = -\infty$$

Example 11 Find $\lim_{x \to \infty}\left(xe^{1/x} - x\right)$

Since $\lim_{x \to \infty} xe^{1/x} = \infty$ and $\lim_{x \to \infty} x = \infty$ the limit is an indeterminate form of type $\infty - \infty$. We factor out a common factor to obtain the limit $\lim_{x \to \infty} x\left(e^{1/x} - 1\right)$ which is an indeterminate form of type $0 \cdot \infty$. We then have to convert it into the quotient $\lim_{x \to \infty} \frac{e^{1/x} - 1}{\frac{1}{x}}$ which is now an indeterminate form of type $\frac{0}{0}$. Using l'Hôpital's Rule, we have:

$$\lim_{x \to \infty}\left(xe^{1/x} - x\right) = \lim_{x \to \infty} x\left(e^{1/x} - 1\right) = \lim_{x \to \infty} \frac{e^{1/x} - 1}{\frac{1}{x}} = \lim_{x \to \infty} \frac{\left(\frac{-1}{x^2}\right)e^{1/x}}{\frac{-1}{x^2}} = \lim_{x \to \infty} e^{1/x} = 1$$

Example 12 Find $\lim_{x \to \infty}(x - \sqrt{x + 2})$

It's easily seen that the given limit is of the indeterminate form of type $\infty - \infty$. We can convert it into an indeterminate form of type $\frac{\infty}{\infty}$ using rationalization:

$$\lim_{x \to \infty}(x - \sqrt{x + 2}) = \lim_{x \to \infty}(x - \sqrt{x + 2})\left(\frac{x + \sqrt{x + 2}}{x + \sqrt{x + 2}}\right) = \lim_{x \to \infty} \frac{x^2 - (x + 2)}{x + \sqrt{x + 2}}$$

And then we apply l'Hôpital's Rule:

$$\lim_{x \to \infty}(x - \sqrt{x + 2}) = \lim_{x \to \infty} \frac{x^2 - (x + 2)}{x + \sqrt{x + 2}} = \lim_{x \to \infty} \frac{2x - 1}{1 + \frac{1}{2\sqrt{x + 2}}} = \infty$$

Example 13 Find $\lim_{x\to 1}\left(\frac{x}{x-1} - \frac{1}{\ln x}\right)$

Since $\lim_{x\to 1}\frac{x}{x-1} = \infty$ and $\lim_{x\to 1}\frac{1}{\ln x} = \infty$ the limit is an indeterminate form of type $\infty - \infty$. We must first convert this using a common denominator:

$$\lim_{x\to 1}\left(\frac{x}{x-1} - \frac{1}{\ln x}\right) = \lim_{x\to 1}\left(\frac{x\ln x}{(x-1)\ln x} - \frac{x-1}{(x-1)\ln x}\right) = \lim_{x\to 1}\frac{x\ln x - x + 1}{(x-1)\ln x}$$

Since $\lim_{x\to 1}(x\ln x - x + 1) = 0$ and $\lim_{x\to 1}(x-1)\ln x = 0$ the limit is now an indeterminate form of type $\frac{0}{0}$. We can therefore apply l'Hôpital's Rule.

$$\lim_{x\to 1}\left(\frac{x}{x-1} - \frac{1}{\ln x}\right) = \lim_{x\to 1}\frac{x\ln x - x + 1}{(x-1)\ln x} = \lim_{x\to 1}\frac{\ln x + \left(\frac{1}{x}\right)x - 1}{\ln x + \left(\frac{1}{x}\right)(x-1)} = \lim_{x\to 1}\frac{\ln x}{\ln x + 1 - \frac{1}{x}}$$

Since the limits of both the numerator and denominator are still 0, we apply l'Hôpital's Rule again.

$$\lim_{x\to 1}\left(\frac{x}{x-1} - \frac{1}{\ln x}\right) = \lim_{x\to 1}\frac{\ln x}{\ln x + 1 - \frac{1}{x}} = \lim_{x\to 1}\frac{\frac{1}{x}}{\frac{1}{x} + \frac{1}{x^2}} = \frac{1}{1+1} = \frac{1}{2}$$

EXERCISES Find each limit. Use L'Hôpital's Rule where appropriate. Otherwise use a more elementary method.

1. $\lim_{x\to 3}\frac{2x^2-3x-9}{x^2-2x-3}$

2. $\lim_{x\to\infty}\frac{4x^3+x-3}{x^2-5x+8}$

3. $\lim_{x\to 1}\frac{2x^3-x^2-4x+3}{3x^3-5x^2+x+1}$

4. $\lim_{x\to\infty}\frac{6x-5}{4x^2+7x+9}$

5. $\lim_{x\to 0}\frac{3x^2+8x}{5x^3}$

6. $\lim_{x\to\infty}\frac{x^2-7x-10}{6x^2-x-1}$

7. $\lim_{x\to 0}\frac{1-e^x}{2x}$

8. $\lim_{x\to 0}\frac{x^2+x}{\sin 3x}$

9. $\lim_{x\to 0^+}\frac{\sin x}{1-\cos x}$

10. $\lim_{x\to 0}\frac{1-\cos x}{\sin x}$

11. $\lim_{x\to 0}\frac{\sin 2x}{\sin x}$

12. $\lim_{x\to 0}\frac{e^x-x-1}{5x^2}$

13. $\lim_{x\to\infty}\frac{e^{3x}}{\ln x}$

14. $\lim_{x\to 1}\frac{x-1}{\sqrt{x}-1}$

15. $\lim_{x\to 0^+}\frac{\sqrt{x}}{1-\cos x}$

16. $\lim_{x\to\infty}\frac{\sqrt{x-1}}{4x+5}$

17. $\lim_{x\to 1}\frac{\ln x}{\sin(x-1)}$

18. $\lim_{x\to\infty}\frac{\ln(x+1)}{\sqrt{x}}$

19. $\lim_{x\to\infty}\frac{e^x+x}{\ln x}$

20. $\lim_{x\to 1}\frac{e^{x-1}-1}{(x-1)^3}$

21. $\lim_{x\to\infty}\frac{\ln(x-10)}{\ln(4x+1)}$

22. $\lim_{x\to 0^+}\frac{\sqrt{x}}{\ln(x+1)}$

23. $\lim_{x\to\infty}\frac{e^{4x}}{e^{3x}+x}$

24. $\lim_{x\to 0}\frac{e^{2x}-1}{e^{5x}-1}$

25. $\lim_{x\to 0^+} x^3\ln x$

26. $\lim_{x\to\infty} x^2e^{-x}$

27. $\lim_{x\to 0^+} x\tan\left(\frac{2}{x}\right)$

28. $\lim_{x\to\infty}\left(x-\sqrt{x^2+1}\right)$

29. $\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)$

30. $\lim_{x\to\infty}\left(\sqrt{x}-x^2\right)$

SOLUTIONS

1. $\lim_{x\to 3}\frac{2x^2-3x-9}{x^2-2x-3}$ $\left(\text{type }\frac{0}{0}\right)$

$= \lim_{x\to 3}\frac{4x-3}{2x-2} = \frac{9}{4}$

2. $\lim_{x\to\infty}\frac{4x^3+x-3}{x^2-5x+8}$ $\left(\text{type }\frac{\infty}{\infty}\right)$

$= \lim_{x\to\infty}\frac{12x^2+1}{2x-5}$ $\left(\text{type }\frac{\infty}{\infty}\right)$

$= \lim_{x\to\infty}\frac{24x}{2} = \infty$

3. $\lim_{x\to 1}\frac{2x^3-x^2-4x+3}{3x^3-5x^2+x+1}$ $\left(\text{type }\frac{0}{0}\right)$

$= \lim_{x\to 1}\frac{6x^2-2x-4}{9x^2-10x+1}$ $\left(\text{type }\frac{0}{0}\right)$

$= \lim_{x\to 1}\frac{12x-2}{18x-10} = \frac{10}{8} = \frac{5}{4}$

4. $\lim_{x\to\infty}\frac{6x-5}{4x^2+7x+9}$ $\left(\text{type }\frac{\infty}{\infty}\right)$

$= \lim_{x\to\infty}\frac{6}{8x+7} = 0$

5. $\lim_{x\to 0}\frac{3x^2+8x}{5x^3}$ $\left(\text{type }\frac{0}{0}\right)$

$= \lim_{x\to 0}\frac{6x+8}{15x^2} = \infty$

6. $\lim_{x\to\infty}\frac{x^2-7x-10}{6x^2-x-1}$ $\left(\text{type }\frac{\infty}{\infty}\right)$

$= \lim_{x\to\infty}\frac{2x-7}{12x-1}$ $\left(\text{type }\frac{\infty}{\infty}\right)$

$= \lim_{x\to\infty}\frac{2}{12} = \frac{2}{12} = \frac{1}{6}$

7. $\lim_{x\to 0}\frac{1-e^x}{2x}$ $\left(\text{type }\frac{0}{0}\right)$

$= \lim_{x\to 0}\frac{-e^x}{2} = -\frac{1}{2}$

8. $\lim_{x\to 0}\frac{x^2+x}{\sin 3x}$ $\left(\text{type }\frac{0}{0}\right)$

$= \lim_{x\to 0}\frac{2x+1}{3\cos 3x} = \frac{1}{3}$

9. $\lim_{x\to 0^+}\frac{\sin x}{1-\cos x}$ $\left(\text{type }\frac{0}{0}\right)$

$= \lim_{x\to 0^+}\frac{\cos x}{\sin x} = \infty$

10. $\lim_{x\to 0}\frac{1-\cos x}{\sin x}$ $\left(\text{type }\frac{0}{0}\right)$

$= \lim_{x\to 0}\frac{\sin x}{\cos x} = 0$

11. $\lim_{x\to 0}\frac{\sin 2x}{\sin x}$ $\left(\text{type }\frac{0}{0}\right)$

$= \lim_{x\to 0}\frac{2\cos 2x}{\cos x} = 2$

12. $\lim_{x\to 0}\frac{e^x-x-1}{5x^2}$ $\left(\text{type }\frac{0}{0}\right)$

$= \lim_{x\to 0}\frac{e^x-1}{10x}$ $\left(\text{type }\frac{0}{0}\right)$

$= \lim_{x\to 0}\frac{e^x}{10} = \frac{1}{10}$

13. $\lim_{x\to\infty} \dfrac{e^{3x}}{\ln x}$ $\left(\text{type } \frac{\infty}{\infty}\right)$

$= \lim_{x\to\infty} \dfrac{\left[3e^{3x}\right]}{\left[\frac{1}{x}\right]}$ $\left(\text{type } \frac{\infty}{\infty}\right)$

$= \lim_{x\to\infty} 3x\,e^{3x} = \infty$

14. $\lim_{x\to 1} \dfrac{x-1}{\sqrt{x}-1}$ $\left(\text{type } \frac{0}{0}\right)$

$= \lim_{x\to 1} \dfrac{[1]}{\left[\frac{1}{2\sqrt{x}}\right]} = 2$

15. $\lim_{x\to 0^+} \dfrac{\sqrt{x}}{1-\cos x}$ $\left(\text{type } \frac{0}{0}\right)$

$= \lim_{x\to 0^+} \dfrac{\left[\frac{1}{2\sqrt{x}}\right]}{[\sin x]}$

$= \lim_{x\to 0^+} \dfrac{1}{2\sqrt{x}\sin x} = \infty$

16. $\lim_{x\to\infty} \dfrac{\sqrt{x-1}}{4x+5}$ $\left(\text{type } \frac{\infty}{\infty}\right)$

$= \lim_{x\to\infty} \dfrac{\left[\frac{1}{2\sqrt{x-1}}\right]}{[4]}$

$= \lim_{x\to\infty} \dfrac{1}{8\sqrt{x-1}} = 0$

17. $\lim_{x\to 1} \dfrac{\ln x}{\sin(x-1)}$ $\left(\text{type } \frac{0}{0}\right)$

$= \lim_{x\to 1} \dfrac{\left[\frac{1}{x}\right]}{[\cos(x-1)]} = 1$

18. $\lim_{x\to\infty} \dfrac{\ln(x+1)}{\sqrt{x}}$ $\left(\text{type } \frac{\infty}{\infty}\right)$

$= \lim_{x\to\infty} \dfrac{\left[\frac{1}{x+1}\right]}{\left[\frac{1}{2\sqrt{x}}\right]}$

$= \lim_{x\to\infty} \dfrac{2\sqrt{x}}{x+1}$ $\left(\text{type } \frac{\infty}{\infty}\right)$

$= \lim_{x\to\infty} \dfrac{\left[\frac{1}{\sqrt{x}}\right]}{[1]} = \lim_{x\to\infty} \dfrac{1}{\sqrt{x}} = 0$

19. $\lim_{x\to\infty} \dfrac{e^{x}+x}{\ln x}$ $\left(\text{type } \frac{\infty}{\infty}\right)$

$= \lim_{x\to\infty} \dfrac{\left[e^{x}+1\right]}{\left[\frac{1}{x}\right]}$

$= \lim_{x\to\infty} x\left(e^{x}+1\right) = \infty$

20. $\lim_{x\to 1} \dfrac{e^{x-1}-1}{(x-1)^3}$ $\left(\text{type } \frac{0}{0}\right)$

$= \lim_{x\to 1} \dfrac{e^{x-1}}{3(x-1)^2} = \infty$

21. $\lim_{x\to\infty} \frac{\ln(x-10)}{\ln(4x+1)}$ (type $\frac{\infty}{\infty}$)

$$= \lim_{x\to\infty} \frac{\left[\frac{1}{x-10}\right]}{\left[\frac{4}{4x+1}\right]}$$

$$= \lim_{x\to\infty} \frac{4x+1}{4x-40} \quad \left(\text{type } \tfrac{\infty}{\infty}\right)$$

$$= \lim_{x\to\infty} \frac{4}{4} = \frac{4}{4} = 1$$

22. $\lim_{x\to 0^+} \frac{\sqrt{x}}{\ln(x+1)}$ (type $\frac{0}{0}$)

$$= \lim_{x\to 0^+} \frac{\left[\frac{1}{2\sqrt{x}}\right]}{\left[\frac{1}{x+1}\right]}$$

$$= \lim_{x\to 0^+} \frac{x+1}{2\sqrt{x}} = \infty$$

23. $\lim_{x\to\infty} \frac{e^{4x}}{e^{3x}+x}$ (type $\frac{\infty}{\infty}$)

$$= \lim_{x\to\infty} \frac{4e^{4x}}{3e^{3x}+1} \quad \left(\text{type } \tfrac{\infty}{\infty}\right)$$

$$= \lim_{x\to\infty} \frac{16e^{4x}}{9e^{3x}} = \lim_{x\to\infty} \frac{16e^{x}}{9} = \infty$$

24. $\lim_{x\to 0} \frac{e^{2x}-1}{e^{5x}-1}$ (type $\frac{0}{0}$)

$$= \lim_{x\to 0} \frac{2e^{2x}}{5e^{5x}} = \lim_{x\to 0} \frac{2}{5e^{3x}} = \frac{2}{5}$$

25. $\lim_{x\to 0^+} x^3 \ln x$ (type $0\cdot\infty$)

$$= \lim_{x\to 0^+} \frac{\ln x}{x^{-3}} \quad \left(\text{type } \tfrac{\infty}{\infty}\right)$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-3x^{-4}}$$

$$= \lim_{x\to 0^+} \frac{x^3}{-3} = 0$$

26. $\lim_{x\to\infty} x^2 e^{-x}$ (type $0\cdot\infty$)

$$= \lim_{x\to\infty} \frac{x^2}{e^x} \quad \left(\text{type } \tfrac{\infty}{\infty}\right)$$

$$= \lim_{x\to\infty} \frac{2x}{e^x} \quad \left(\text{type } \tfrac{\infty}{\infty}\right)$$

$$= \lim_{x\to\infty} \frac{2}{e^x} = 0$$

27. $\lim_{x\to 0^+} x\tan\left(\frac{2}{x}\right)$ (type $0\cdot\infty$)

$$= \lim_{x\to 0^+} \frac{\tan\left(\frac{2}{x}\right)}{\frac{1}{x}} \quad \left(\text{type } \tfrac{\infty}{\infty}\right)$$

$$= \lim_{x\to 0^+} \frac{\left(\sec^2\left(\frac{2}{x}\right)\right)(-2x^{-2})}{-x^{-2}}$$

$$= \lim_{x\to 0^+} 2\sec^2\left(\frac{2}{x}\right) = 2$$

28. $\lim_{x\to\infty}\left(x - \sqrt{x^2+1}\right)$ (type $\infty - \infty$)

$$= \lim_{x\to\infty} \frac{(x-\sqrt{x^2+1})}{1}\cdot\frac{(x+\sqrt{x^2+1})}{(x+\sqrt{x^2+1})}$$

$$= \lim_{x\to\infty} \frac{x^2-(x^2+1)}{x+\sqrt{x^2+1}}$$

$$= \lim_{x\to\infty} \frac{-1}{x+\sqrt{x^2+1}} = 0$$

29. $\lim_{x\to 0^+}\left(\frac{1}{x} - \frac{1}{e^x-1}\right)$ (type ∞ − ∞)

$= \lim_{x\to 0^+}\frac{(e^x-1)-x}{x(e^x-1)}$ (type $\frac{0}{0}$)

$= \lim_{x\to 0^+}\frac{e^x-1}{1(e^x-1)+(e^x)x}$ (type $\frac{0}{0}$)

$= \lim_{x\to 0^+}\frac{e^x}{e^x+(e^x)x+1(e^x)}$

$= \lim_{x\to 0^+}\frac{e^x}{e^x(2+x)}$

$= \lim_{x\to 0^+}\frac{1}{(2+x)} = \frac{1}{2}$

30. $\lim_{x\to\infty}\left(\sqrt{x} - x^2\right)$ (type ∞ − ∞)

$= \lim_{x\to\infty}\sqrt{x}\left(1 - x^{3/2}\right) = -\infty$