Dawson College

Mathematics Department

FINAL EXAMINATION

Linear Algebra and Vector Geometry for Social Science

201-MA3-DW sections 00001, 00002

June 3rd , 2024

Student Name:———————————

Student I.D#:———————————

Instructors: Oxana Cerba, Gilbert Honnonuovo

Time: 9:30 - 12:30

Instructions:

- Print your name and student ID number in the space provided on the Cover Sheet.
- All questions are to be answered directly on the examination paper in the space provided. Show your complete work and give explanations.
- ONLY SHARP EL-531X*** are permitted.

This examination consists of 14 **questions. Please ensure that you have a complete examination.**

This examination must be returned intact.

Question #1. [6+2 points] Solve using Gaussian Elimination with backward substitution. (Show all your steps)

$$\begin{cases} x_1 + x_2 + 3x_3 - 3x_4 = 2 \\ -x_1 + x_2 + x_3 = 1 \\ x_1 + x_2 + 5x_3 - 3x_4 = 6 \\ x_1 - x_2 + x_3 = 3 \end{cases}$$

and find the particular solution for $x_1 = 3$

Question #2. [3+3 points] For what value(s) of k the following system has (a)no solution, (b)unique solution, (c) infinitely many solutions?

$$\begin{cases} x - y + 4z = 2 \\ kx + 4z = 1 \\ y - kz = -k \end{cases}$$

Question #3. [5+2 points] Consider $A = \begin{bmatrix} 4 & 0 & 2 \\ 2 & -1 & -4 \\ 5 & -2 & -9 \end{bmatrix}$ and the system $\begin{cases} 4x + 2z = 8 \\ 2x - y - 4z = -7 \\ 5x - 2y - 9z = -15 \end{cases}$

(a) Find A^{-1} using the adjoint matrix.

(b) Use your answer in (a) to solve the system.

Question #4. [5 points] Assuming A,B and C are invertible matrices, simplify completely the following expression $(3AB^{-1})^{-1}(B^{-1}A^T)^T(\frac{1}{2}C^TB)^{-1}C^T$

Question #5. [5 points] Given $A^4 + 2A^3 - 2A + I = 0$. Show that A is invertible and find A^{-1} in terms of A and I only.

Question #6.[8 points] Find a matrix A if

$$\left(3A^T - 3\begin{bmatrix}-1 & 2\\ -2 & 3\end{bmatrix}\right)^T = \left(\begin{bmatrix}5 & 3\\ 3 & 2\end{bmatrix}\right)^{-1}$$

Question #7. [3+3+4 points] If C and D are 3 by 3 martices and $det(C) = 4$ and $det(D) = -2$ then compute

(a) $det(det(D)(3D)^{-1})$

(b) $det(C^{-1} - adj(C))$

(c) Find $det(X)$ if $det(2C^2(4CDX)^{-1}) = \frac{1}{4}$

Question #8. [2+2+4 points]

Given three vectors $\mathbf{u} =< 2, -1, 0 >$, $\mathbf{v} =< -2, 1, 1 >$, $\mathbf{w} =< -1, 1, 2 >$ and θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

(a) Find $\|2\mathbf{v} + \mathbf{w}\|$

(b) Find $\cos\theta$

(c) Solve the following equation for $\mathbf{x}$

$$\sqrt{2}\|2\mathbf{v} + \mathbf{w}\|\mathbf{u} + (\mathbf{u} \cdot \mathbf{v})\mathbf{x} = \sqrt{6}\|5\mathbf{w}\|\mathbf{v} + \sqrt{30}\cos\theta\mathbf{w}$$

Question #9. [4+3 points] Given a point P(5,2,1) and two planes $pl_1 : 2x-y+z=0$ and $pl_2 : 2x-2y-z=0$.

(a) Find the parametric equations of the line passing through the point P(5,2,1) and parallel to two planes.

(b) Find the general equation of the plane that contains the point P and orthogonal to the line obtained in part (a)

Question #10. [3+3+3 points] Let $\mathbf{u} =< -1, 2, 1 >$, $\mathbf{v} =< 2, 0, 1 >$ and $\mathbf{w} =< -1, 3, 2 >$. Find

(a) the value of k such that $k\mathbf{u} + \mathbf{v}$ is perpendicular to $\mathbf{w}$

(b) the volume of parallelepiped with edges $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$

(c) the equation of the line orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ through the origin.

Question #11. [2+5 points] Consider the lines $\mathcal{L}_1 : x = -1 + 2t,\ y = 3 - t,\ z = t$ and $\mathcal{L}_2 : x = -1 - 4s,\ y = 3 + 2s,\ z = 3 - 2s.$

(a) Show that $\mathcal{L}_1$ and $\mathcal{L}_2$ are parallel.

(b) Find the equation for the plane containing $\mathcal{L}_1$ and $\mathcal{L}_2$

Question #12. [**7 points**] Maximize $Z = 2x - y + 4z$ subject to the constrains

$$\begin{cases} y + z \leq 30 \\ 2x - y + 2z \leq 80 \\ 2x - y - z \leq 10 \end{cases}$$

Question #13. [7 points] An economy has two sectors: Electricity and Services. For each unit of output, Electricity requires 0.5 units from its own sector and 0.4 units from Services. Meanwhile, Services requires 0.5 units from Electricity and 0.2 units from its own sector to produce one unit of Services. Use an inverse matrix to determine the production vector necessary to satisfy a final demand of 1000 units of Electricity and 2000 units of Services.

Question #14. [2+4 points] Suppose there are two states (think countries, or US states, or cities, or whatever) 1 and 2 with a total population of 1 distributed as 0.7 in State 1 and 0.3 in State 2. Suppose that at the end of the year 10% of the people in State 1 move out of State 1 and into State 2 (the rest remain) and 5% of the people in State 2 move out of State 2 and into State 1 (the rest remain).

In the other words you are given transition matrix $A = \begin{bmatrix} 0.9 & 0.05 \\ 0.1 & 0.95 \end{bmatrix}$ and initial state vector $x^{(0)} = \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix}$

(a) What will the population distribution be in one year, two years, three years? (i.e. find $x^{(1)}, x^{(2)}, x^{(3)}$)

(b) Suppose this keeps happening year-by-year for years, so what happens in the long term? (i.e. find the steady-state vector q and interpret it in given context)

ANSWERS

Question #1.

General solution $(x_1, x_2, x_3, x_4) = (-3/2 + 3/2t, -5/2 + 3/2t, 2, t)$

Particular solution $(x_1, x_2, x_3, x_4) = (3, 2, 2, 3)$

Question #2.

(a) No solution when $k = 2$, (b) Unique solution when $k \neq 2$, (c) infinitely many solutions in not possible in this case.

Question #3.

(a) $A^{-1} = 1/6 \begin{bmatrix} 1 & -4 & 2 \\ -2 & -46 & 20 \\ 1 & 8 & -4 \end{bmatrix}$

(b) $(x, y, z) = (1, 1, 2)$

Question #4.

$2/3B(B^T)B^{-1}$

Question #5.

$A^{-1} = -A^3 - 2A^2 + 2I$

Question #6. $A = \begin{bmatrix} -1/3 & -3 \\ 1 & 14/3 \end{bmatrix}$

Question #7.

(a) 4/27, (b) -27/4 (c) -1

Question #8.

(a)$\sqrt{50}$, (b) -5/$\sqrt{30}$ (c) $x = (15, -7, -4)$

Question #9.

(a)Line $(x, y, z) = (5 + 3t, 2 + 4t, 1 - 2t)$, (b) Plane $3x - 4y - 2z = 21$

Question #10.

(a) k=0, (b) Volume=1 (c) Line $(x, y, z) = (2t, 3t, -4t)$

Question #11.

(a) Proportional. (b) Plane $x + 2y = 5$

Question #12.

Solution $(x, y, z) = (10, 0, 30)$ Max P=140

Question #13.

(9000,7000)

Question #14.

(a) $x^{(1)} = (0.645, 0.355), x^{(2)} = (0.59825, 0.40175), x^{(3)} = (0.5585125, 0.4414875)$ (b) Steady state= (1/3, 2/3)